The Inverse z-Transform

In science one tries to tell people, in such a way as to be understood by everyone, something that no one ever knew before. But in poetry, it's the exact opposite.

Paul Dirac
The Inverse Z-Transform

• Formal inverse z-transform is based on a Cauchy integral
• Less formal ways sufficient most of the time
  – Inspection method
  – Partial fraction expansion
  – Power series expansion

• Inspection Method
  – Make use of known z-transform pairs such as

\[a^n u[n] \leftrightarrow z \rightarrow \frac{1}{1 - az^{-1}} \quad |z| > |a|\]

– Example: The inverse z-transform of

\[X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} \quad |z| > \frac{1}{2} \quad \rightarrow \quad x[n] = \left(\frac{1}{2}\right)^n u[n]\]
Inverse Z-Transform by Partial Fraction Expansion

- Assume that a given z-transform can be expressed as
  \[ X(z) = \sum_{k=0}^{M} b_k z^{-k} \sum_{k=0}^{N} a_k z^{-k} \]

- Apply partial fractional expansion
  \[ X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1, k \neq i}^{N} \frac{A_k}{1 - d_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{(1 - d_i z^{-1})^m} \]

- First term exist only if \( M > N \)
  - \( B_r \) is obtained by long division
- Second term represents all first order poles
- Third term represents an order \( s \) pole
  - There will be a similar term for every high-order pole
- Each term can be inverse transformed by inspection
Partial Fractional Expression

\[ X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1, k \neq i}^{N} \frac{A_k}{1 - d_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{(1 - d_i z^{-1})^m} \]

- Coefficients are given as

\[ A_k = (1 - d_k z^{-1})X(z) \bigg|_{z=d_k} \]

\[ C_m = \frac{1}{(s - m)! (-d_i)^{s-m}} \left\{ \frac{d^{s-m}}{dw^{s-m}} \left[ (1 - d_i w)^s X(w^{-1}) \right] \right\}_{w=d_i^{-1}} \]

- Easier to understand with examples
Example: 2\textsuperscript{nd} Order Z-Transform

\[ X(z) = \frac{1}{\left(1 - \frac{1}{4} z^{-1}\right)\left(1 - \frac{1}{2} z^{-1}\right)} \quad \text{ROC: } |z| > \frac{1}{2} \]

- Order of nominator is smaller than denominator (in terms of \(z^{-1}\))
- No higher order pole

\[ X(z) = \frac{A_1}{1 - \frac{1}{4} z^{-1}} + \frac{A_2}{1 - \frac{1}{2} z^{-1}} \]

\[ A_1 = \left(1 - \frac{1}{4} z^{-1}\right)X(z) \bigg|_{z=\frac{1}{4}} = \frac{1}{\left(1 - \frac{1}{4} \left(\frac{1}{4}\right)^{-1}\right)} = -1 \]

\[ A_2 = \left(1 - \frac{1}{2} z^{-1}\right)X(z) \bigg|_{z=\frac{1}{2}} = \frac{1}{\left(1 - \frac{1}{2} \left(\frac{1}{2}\right)^{-1}\right)} = 2 \]
Example Continued

\[ X(z) = \frac{-1}{\left(1 - \frac{1}{4} z^{-1}\right)} + \frac{2}{\left(1 - \frac{1}{2} z^{-1}\right)} \quad |z| > \frac{1}{2} \]

- ROC extends to infinity
  - Indicates right sided sequence

\[ x[n] = 2 \left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)^n u[n] \]
Example #2

\[
X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = \frac{(1 + z^{-1})^2}{(1 - \frac{1}{2}z^{-1})(1 - z^{-1})} \quad |z| > 1
\]

• Long division to obtain \(B_o\)

\[
\frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 = \frac{2}{z^{-2} + 2z^{-1} + 1} = \frac{2}{z^{-2} - 3z^{-1} + 2} \quad z^{-2} - 3z^{-1} + 2 = 5z^{-1} - 1
\]

\[
A_1 = \left(1 - \frac{1}{2}z^{-1}\right)X(z) \bigg|_{z = \frac{1}{2}} = -9
\]

\[
A_2 = (1 - z^{-1})X(z) \bigg|_{z = 1} = 8
\]
Example #2 Continued

\[ X(z) = 2 - \frac{9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{1 - z^{-1}} \quad |z| > 1 \]

- ROC extends to infinity
  - Indicates right-sides sequence

\[ x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] - 8u[n] \]
Inverse Z-Transform by Power Series Expansion

- The z-transform is power series
  \[ X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \]
- In expanded form
  \[ X(z) = \cdots + x[-2] z^2 + x[-1] z^1 + x[0] + x[1] z^{-1} + x[2] z^{-2} + \cdots \]
- Z-transforms of this form can generally be inverted easily
- Especially useful for finite-length series
- Example
  \[ X(z) = z^2 \left(1 - \frac{1}{2} z^{-1}\right)(1 + z^{-1})(1 - z^{-1}) \]
  \[ = z^2 - \frac{1}{2} z - 1 + \frac{1}{2} z^{-1} \]
  \[ x[n] = \delta[n + 2] - \frac{1}{2} \delta[n + 1] - \delta[n] + \frac{1}{2} \delta[n - 1] \]
  \[ x[n] = \begin{cases} 
  1 & n = -2 \\
  -1 & n = -1 \\
  1 & n = 0 \\
  1 & n = 1 \\
  2 & n = 2 
\end{cases} \]
\[ x[n] = \begin{cases} 
    a^n & 0 \leq n \leq N - 1 \\
    0 & \text{otherwise}
\end{cases} \]

\[ X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n = \frac{1 - (az^{-1})^N}{1 - az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z - a} \]
Z-Transform Properties: Linearity

- Notation

\[ x[n] \overset{z}{\longleftrightarrow} X(z) \quad \text{ROC} = R_x \]

- Linearity

\[ ax_1[n] + bx_2[n] \overset{z}{\longleftrightarrow} aX_1(z) + bX_2(z) \quad \text{ROC} = R_{x_1} \cap R_{x_2} \]

- Note that the ROC of combined sequence may be larger than either ROC
- This would happen if some pole/zero cancellation occurs
- Example:

\[ x[n] = a^n u[n] - a^n u[n - N] \]

- Both sequences are right-sided
- Both sequences have a pole \( z = a \)
- Both have a ROC defined as \( |z| > |a| \)
- In the combined sequence the pole at \( z = a \) cancels with a zero at \( z = a \)
- The combined ROC is the entire \( z \) plane except \( z = 0 \)

- We did make use of this property already, where?
Z-Transform Properties: Time Shifting

\[ x[n - n_0] \xrightarrow{z} z^{-n_0} X(z) \quad \text{ROC} = R_x \]

- Here \( n_0 \) is an integer
  - If positive the sequence is shifted right
  - If negative the sequence is shifted left
- The ROC can change the new term may
  - Add or remove poles at \( z=0 \) or \( z=\infty \)
- Example

\[
X(z) = z^{-1} \left( \frac{1}{1 - \frac{1}{4} z^{-1}} \right) \quad |z| > \frac{1}{4}
\]

\[
x[n] = \left( \frac{1}{4} \right)^{n-1} u[n - 1]
\]
Z-Transform Properties: Multiplication by Exponential

\[ z_0^n x[n] \xrightarrow{z} X(z / z_0) \quad \text{ROC} = |z_0| R_x \]

- ROC is scaled by |z₀|
- All pole/zero locations are scaled
- If z₀ is a positive real number: z-plane shrinks or expands
- If z₀ is a complex number with unit magnitude it rotates
- Example: We know the z-transform pair

\[ u[n] \xrightarrow{z} \frac{1}{1 - z^{-1}} \quad \text{ROC} : |z| > 1 \]

- Let’s find the z-transform of

\[ x[n] = r^n \cos(\omega_0 n) u[n] = \frac{1}{2} (r e^{j\omega_0})^n u[n] + \frac{1}{2} (r^{-j\omega_0})^n u[n] \]

\[ X(z) = \frac{1/2}{1 - re^{j\omega_0}z^{-1}} + \frac{1/2}{1 - re^{-j\omega_0}z^{-1}} \quad |z| > r \]
Z-Transform Properties: Differentiation

\[ nx[n] \xleftarrow{z} -z \frac{dX(z)}{dz} \quad \text{ROC} = R_x \]

- Example: We want the inverse z-transform of
  \[ X(z) = \log(1 + az^{-1}) \quad |z| > |a| \]
- Let’s differentiate to obtain rational expression
  \[ \frac{dX(z)}{dz} = -az^{-2} \quad \Rightarrow \quad -z \frac{dX(z)}{dz} = az^{-1} \frac{1}{1 + az^{-1}} \]
- Making use of z-transform properties and ROC
  \[ nx[n] = a(-a)^{n-1} u[n - 1] \]
  \[ x[n] = (-1)^{n-1} \frac{a^n}{n} u[n - 1] \]
Z-Transform Properties: Conjugation

\[ x^*[n] \xrightarrow{z} X^*(z^*) \quad \text{ROC} = R_x \]

- Example

\[
X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}
\]

\[
X^*(z) = \left( \sum_{n=-\infty}^{\infty} x[n] z^{-n} \right)^* = \sum_{n=-\infty}^{\infty} x^*[n] z^n
\]

\[
X^*(z^*) = \sum_{n=-\infty}^{\infty} x^*[n] (z^n)^* = \sum_{n=-\infty}^{\infty} x^*[n] z^{-n} = Z\{x^*[n]\}
\]
Z-Transform Properties: Time Reversal

\[ x[-n] \xrightarrow{z} X(1/z) \quad \text{ROC} = \frac{1}{R_x} \]

- ROC is inverted
- Example:

\[ x[n] = a^{-n}u[-n] \]

- Time reversed version of \( a^n u[n] \)

\[ X(z) = \frac{1}{1 - az} = \frac{-a^{-1}z^{-1}}{1 - a^{-1}z^{-1}} \quad |z| < |a^{-1}| \]
Z-Transform Properties: Convolution

\[ x_1[n] \ast x_2[n] \xrightarrow{\text{z}} X_1(z)X_2(z) \quad \text{ROC : } R_{x_1} \cap R_{x_2} \]

\begin{itemize}
    
    \item Convolution in time domain is multiplication in z-domain
    
    \item Example: Let’s calculate the convolution of

    \[ x_1[n] = a^n u[n] \; \text{and} \; x_2[n] = u[n] \]

    \[ X_1(z) = \frac{1}{1 - az^{-1}} \quad \text{ROC : } |z| > |a| \quad \text{and} \quad X_2(z) = \frac{1}{1 - z^{-1}} \quad \text{ROC : } |z| > 1 \]

    \item Multiplications of z-transforms is

    \[ Y(z) = X_1(z)X_2(z) = \frac{1}{(1 - az^{-1})(1 - z^{-1})} \]

    \item ROC: if \(|a| < 1\) ROC is \(|z| > 1\) if \(|a| > 1\) ROC is \(|z| > |a|\)

    \item Partial fractional expansion of \(Y(z)\)

    \[ Y(z) = \frac{1}{1 - a} \left( \frac{1}{1 - z^{-1}} - \frac{1}{1 - az^{-1}} \right) \quad \text{assume ROC : } |z| > 1 \]

    \[ y[n] = \frac{1}{1 - a} \left( u[n] - a^{n+1}u[n] \right) \]

\end{itemize}