4.117 A vertical force $P$ of magnitude 20 kips is applied at point $C$ located on the axis of symmetry of the cross section of a short column. Knowing that $y = 5$ in., determine (a) the stress at point $A$, (b) the stress at point $B$, (c) the location of the neutral axis.

4.118 A vertical force $P$ is applied at point $C$ located on the axis of symmetry of the cross section of a short column. Determine the range of values of $y$ for which tensile stresses do not occur in the column.

4.119 Knowing that the clamp shown has been tightened until $P = 400$ N, determine (a) the stress at point $A$, (b) the stress at point $B$, (c) the location of the neutral axis of section $a-a$.

4.120 The four bars shown have the same cross-sectional area. For the given loadings, show that (a) the maximum compressive stresses are in the ratio 4:5:7:9, (b) the maximum tensile stresses are in the ratio 2:3:5:3. (Note: the cross section of the triangular bar is an equilateral triangle.)
4.121 The C-shaped steel bar is used as a dynamometer to determine the magnitude \( P \) of the forces shown. Knowing that the cross section of the bar is a square of side 40 mm and that the strain on the inner edge was measured and found to be 450 \( \mu \), determine the magnitude \( P \) of the forces. Use \( E = 200 \text{ GPa} \).

4.122 An eccentric force \( P \) is applied as shown to a steel bar of 25 \( \times \) 90-mm cross section. The strains at \( A \) and \( B \) have been measured and found to be

\[
\epsilon_A = +350 \mu \quad \epsilon_B = -70 \mu
\]

Knowing that \( E = 200 \text{ GPa} \), determine (a) the distance \( d \), (b) the magnitude of the force \( P \).

4.123 Solve Prob. 4.122, assuming that the measured strains are

\[
\epsilon_A = +420 \mu \quad \epsilon_B = +420 \mu
\]

4.124 A short length of a W8 \( \times \) 31 rolled-steel shape supports a rigid plate on which two loads \( P \) and \( Q \) are applied as shown. The strains at two points \( A \) and \( B \) on the centerline of the outer faces of the flanges have been measured and found to be

\[
\epsilon_A = -550 \times 10^{-6} \text{ in./in.} \quad \epsilon_B = -680 \times 10^{-6} \text{ in./in.}
\]

Knowing that \( E = 29 \times 10^6 \text{ psi} \), determine the magnitude of each load.

4.125 Solve Prob. 4.124, assuming that the measured strains are

\[
\epsilon_A = +35 \times 10^{-6} \text{ in./in.} \quad \epsilon_B = -450 \times 10^{-6} \text{ in./in.}
\]

4.126 The eccentric axial force \( P \) acts at point \( D \), which must be located 25 mm below the top surface of the steel bar shown. For \( P = 60 \text{ kN} \), determine (a) the depth \( d \) of the bar for which the tensile stress at point \( A \) is maximum, (b) the corresponding stress at point \( A \).
Our analysis of pure bending has been limited so far to members possessing at least one plane of symmetry and subjected to couples acting in that plane. Because of the symmetry of such members and of their loadings, we concluded that the members would remain symmetric with respect to the plane of the couples and thus bend in that plane (Sec. 4.3). This is illustrated in Fig. 4.49; part a shows the cross section of a member possessing two planes of symmetry, one vertical and one horizontal, and part b the cross section of a member with a single, vertical plane of symmetry. In both cases the couple exerted on the section acts in the vertical plane of symmetry of the member and is represented by the horizontal couple vector \( M \), and in both cases the neutral axis of the cross section is found to coincide with the axis of the couple.

Let us now consider situations where the bending couples do not act in a plane of symmetry of the member, either because they act in a different plane, or because the member does not possess any plane of symmetry. In such situations, we cannot assume that the member will bend in the plane of the couples. This is illustrated in Fig. 4.50. In each part of the figure, the couple exerted on the section has again been assumed to act in a vertical plane and has been represented by a horizontal couple vector \( M \). However, since the vertical plane is not a plane of symmetry, we cannot expect the member to bend in that plane, or the neutral axis of the section to coincide with the axis of the couple.

We propose to determine the precise conditions under which the neutral axis of a cross section of arbitrary shape coincides with the axis of the couple \( M \) representing the forces acting on that section. Such a section is shown in Fig. 4.51, and both the couple vector \( M \) and the

\[
\sigma_{xx} \, dA = M \cos \theta \, dx
\]
neutral axis have been assumed to be directed along the \( z \) axis. We recall from Sec. 4.2 that, if we then express that the elementary internal forces \( \sigma_x dA \) form a system equivalent to the couple \( \mathbf{M} \), we obtain

\[
\begin{align*}
\text{x components:} & \quad \int \sigma_x dA = 0 \quad (4.1) \\
\text{moments about y axis:} & \quad \int z \sigma_x dA = 0 \quad (4.2) \\
\text{moments about z axis:} & \quad \int (-y \sigma_x dA) = M \quad (4.3)
\end{align*}
\]

As we saw earlier, when all the stresses are within the proportional limit, the first of these equations leads to the requirement that the neutral axis be a centroidal axis, and the last to the fundamental relation \( \sigma_x = -\frac{M y}{I} \). Since we had assumed in Sec. 4.2 that the cross section was symmetric with respect to the \( y \) axis, Eq. (4.2) was dismissed as trivial at that time. Now that we are considering a cross section of arbitrary shape, Eq. (4.2) becomes highly significant. Assuming the stresses to remain within the proportional limit of the material, we can substitute \( \sigma_x = -\sigma_m y/c \) into Eq. (4.2) and write

\[
\int z \left( -\frac{\sigma_m y}{c} \right) dA = 0 \quad \text{or} \quad \int y z dA = 0 \quad (4.51)
\]

The integral \( \int y z dA \) represents the product of inertia \( I_{yz} \) of the cross section with respect to the \( y \) and \( z \) axes, and will be zero if these axes are the principal centroidal axes of the cross section.† We thus conclude that the neutral axis of the cross section will coincide with the axis of the couple \( \mathbf{M} \) representing the forces acting on that section if, and only if, the couple vector \( \mathbf{M} \) is directed along one of the principal centroidal axes of the cross section.

We note that the cross sections shown in Fig. 4.49 are symmetric with respect to at least one of the coordinate axes. It follows that, in each case, the \( y \) and \( z \) axes are the principal centroidal axes of the section. Since the couple vector \( \mathbf{M} \) is directed along one of the principal centroidal axes, we verify that the neutral axis will coincide with the axis of the couple. We also note that, if the cross sections are rotated through 90° (Fig. 4.52), the couple vector \( \mathbf{M} \) will still be directed along a principal centroidal axis, and the neutral axis will again coincide with the axis of the couple, even though in case \( b \) the couple does not act in a plane of symmetry of the member.

In Fig. 4.50, on the other hand, neither of the coordinate axes is an axis of symmetry for the sections shown, and the coordinate axes are not principal axes. Thus, the couple vector \( \mathbf{M} \) is not directed along a principal centroidal axis, and the neutral axis does not coincide with the axis of the couple. However, any given section possesses principal centroidal axes, even if it is unsymmetric, as the section shown in Fig. 4.50\(c\), and these axes may be determined analytically or by using Mohr’s circle.† If the couple vector \( \mathbf{M} \) is directed along one of the principal centroidal axes of the section, the neutral axis will coincide with the axis of the couple (Fig. 4.53) and the equations

derived in Secs. 4.3 and 4.4 for symmetric members can be used to determine the stresses in this case as well.

As you will see presently, the principle of superposition can be used to determine stresses in the most general case of unsymmetric bending. Consider first a member with a vertical plane of symmetry, which is subjected to bending couples \( M \) and \( M' \) acting in a plane forming an angle \( \theta \) with the vertical plane (Fig. 4.54). The couple vector \( M \) representing the forces acting on a given cross section will form the same angle \( \theta \) with the horizontal \( z \) axis (Fig. 4.55). Resolving the vector \( M \) into component vectors \( M_z \) and \( M_y \) along the \( z \) and \( y \) axes, respectively, we write

\[
M_z = M \cos \theta \quad M_y = M \sin \theta
\]

(4.52)

Since the \( y \) and \( z \) axes are the principal centroidal axes of the cross section, we can use Eq. (4.16) to determine the stresses resulting from the application of either of the couples represented by \( M_z \) and \( M_y \). The couple \( M_z \) acts in a vertical plane and bends the member in that plane (Fig. 4.56). The resulting stresses are

\[
\sigma_x = -\frac{M_z y}{I_z}
\]

(4.53)

where \( I_z \) is the moment of inertia of the section about the principal centroidal \( z \) axis. The negative sign is due to the fact that we have compression above the \( xz \) plane (\( y > 0 \)) and tension below (\( y < 0 \)). On the other hand, the couple \( M_y \) acts in a horizontal plane and bends the member in that plane (Fig. 4.57). The resulting stresses are found to be

\[
\sigma_x = +\frac{M_y z}{I_y}
\]

(4.54)

where \( I_y \) is the moment of inertia of the section about the principal centroidal \( y \) axis, and where the positive sign is due to the fact that we have tension to the left of the vertical \( xy \) plane (\( z > 0 \)) and compression to its right (\( z < 0 \)). The distribution of the stresses caused by the original couple \( M \) is obtained by superposing the stress distributions defined by Eqs. (4.53) and (4.54), respectively. We have

\[
\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}
\]

(4.55)
We note that the expression obtained can also be used to compute the stresses in an unsymmetric section, such as the one shown in Fig. 4.58, once the principal centroidal $y$ and $z$ axes have been determined. On the other hand, Eq. (4.55) is valid only if the conditions of applicability of the principle of superposition are met. In other words, it should not be used if the combined stresses exceed the proportional limit of the material, or if the deformations caused by one of the component couples appreciably affect the distribution of the stresses due to the other.

Equation (4.55) shows that the distribution of stresses caused by unsymmetric bending is linear. However, as we have indicated earlier in this section, the neutral axis of the cross section will not, in general, coincide with the axis of the bending couple. Since the normal stress is zero at any point of the neutral axis, the equation defining that axis can be obtained by setting $\sigma_x = 0$ in Eq. (4.55). We write

$$\frac{-M_y y}{I_z} + \frac{M_z z}{I_y} = 0$$

or, solving for $y$ and substituting for $M_z$ and $M_y$ from Eqs. (4.52),

$$y = \left(\frac{I_z}{I_y} \tan \theta\right) z \quad (4.56)$$

The equation obtained describes a straight line of slope $m = (I_z/I_y) \tan \theta$. Thus, the angle $\phi$ that the neutral axis forms with the $z$ axis (Fig. 4.59) is defined by the relation

$$\tan \phi = \frac{I_z}{I_y} \tan \theta \quad (4.57)$$

where $\theta$ is the angle that the couple vector $\mathbf{M}$ forms with the same axis. Since $I_z$ and $I_y$ are both positive, $\phi$ and $\theta$ have the same sign. Furthermore, we note that $\phi > \theta$ when $I_z > I_y$, and $\phi < \theta$ when $I_z < I_y$. Thus, the neutral axis is always located between the couple vector $\mathbf{M}$ and the principal axis corresponding to the minimum moment of inertia.
A 1600-lb · in. couple is applied to a wooden beam, of rectangular cross section 1.5 by 3.5 in., in a plane forming an angle of 30° with the vertical (Fig. 4.60). Determine (a) the maximum stress in the beam, (b) the angle that the neutral surface forms with the horizontal plane.

(a) Maximum Stress. The components $M_x$ and $M_y$ of the couple vector are first determined (Fig. 4.61):

$$M_x = (1600 \text{ lb \cdot in.}) \cos 30° = 1386 \text{ lb \cdot in.}$$
$$M_y = (1600 \text{ lb \cdot in.}) \sin 30° = 800 \text{ lb \cdot in.}$$

We also compute the moments of inertia of the cross section with respect to the $z$ and $y$ axes:

$$I_z = \frac{1}{12}(1.5 \text{ in.})(3.5 \text{ in.})^3 = 5.359 \text{ in}^4$$
$$I_y = \frac{1}{12}(3.5 \text{ in.})(1.5 \text{ in.})^3 = 0.9844 \text{ in}^4$$

The largest tensile stress due to $M_x$ occurs along $AB$ and is

$$\sigma_1 = \frac{M_y}{I_z} = \frac{(1386 \text{ lb \cdot in.})(1.75 \text{ in.})}{5.359 \text{ in}^4} = 452.6 \text{ psi}$$

The largest tensile stress due to $M_y$ occurs along $AD$ and is

$$\sigma_2 = \frac{M_x}{I_y} = \frac{(800 \text{ lb \cdot in.})(0.75 \text{ in.})}{0.9844 \text{ in}^4} = 609.5 \text{ psi}$$

The largest tensile stress due to the combined loading, therefore, occurs at $A$ and is

$$\sigma_{\text{max}} = \sigma_1 + \sigma_2 = 452.6 + 609.5 = 1062 \text{ psi}$$

The largest compressive stress has the same magnitude and occurs at $E$.

(b) Angle of Neutral Surface with Horizontal Plane. The angle $\phi$ that the neutral surface forms with the horizontal plane (Fig. 4.62) is obtained from Eq. (4.57):

$$\tan \phi = \frac{I_z}{I_y} \tan \theta = \frac{5.359 \text{ in}^4}{0.9844 \text{ in}^4} \tan 30° = 3.143$$
$$\phi = 72.4°$$

The distribution of the stresses across the section is shown in Fig. 4.63.
4.14 GENERAL CASE OF ECCENTRIC AXIAL LOADING

In Sec. 4.12 you analyzed the stresses produced in a member by an eccentric axial load applied in a plane of symmetry of the member. You will now study the more general case when the axial load is not applied in a plane of symmetry.

Consider a straight member \( AB \) subjected to equal and opposite eccentric axial forces \( P \) and \( P' \) (Fig. 4.64\( a \)), and let \( a \) and \( b \) denote the distances from the line of action of the forces to the principal centroidal axes of the cross section of the member. The eccentric force \( P \) is statically equivalent to the system consisting of a centric force \( P \) and of the two couples \( M_y \) and \( M_z \) of moments \( M_y = Pa \) and \( M_z = Pb \) represented in Fig. 4.64\( b \). Similarly, the eccentric force \( P' \) is equivalent to the centric force \( P' \) and the couples \( M'_y \) and \( M'_z \).

By virtue of Saint-Venant’s principle (Sec. 2.17), we can replace the original loading of Fig. 4.64\( a \) by the statically equivalent loading of Fig. 4.64\( b \) in order to determine the distribution of stresses in a section \( S \) of the member, as long as that section is not too close to either end of the member. Furthermore, the stresses due to the loading of Fig. 4.64\( b \) can be obtained by superposing the stresses corresponding to the centric axial load \( P \) and to the bending couples \( M_y \) and \( M_z \), as long as the conditions of applicability of the principle of superposition are satisfied (Sec. 2.12). The stresses due to the centric load \( P \) are given by Eq. (1.5), and the stresses due to the bending couples by Eq. (4.55), since the corresponding couple vectors are directed along the principal centroidal axes of the section.

We write, therefore,

\[
\sigma_x = \frac{P}{A} - \frac{M_y y}{I_z} + \frac{M_z z}{I_y}
\]  

(4.58)

where \( y \) and \( z \) are measured from the principal centroidal axes of the section. The relation obtained shows that the distribution of stresses across the section is **linear**.

In computing the combined stress \( \sigma_x \) from Eq. (4.58), care should be taken to correctly determine the sign of each of the three terms in the right-hand member, since each of these terms can be positive or negative, depending upon the sense of the loads \( P \) and \( P' \) and the location of their line of action with respect to the principal centroidal axes of the cross section. Depending upon the geometry of the cross section and the location of the line of action of \( P \) and \( P' \), the combined stresses \( \sigma_x \) obtained from Eq. (4.58) at various points of the section may all have the same sign, or some may be positive and others negative. In the latter case, there will be a line in the section, along which the stresses are zero. Setting \( \sigma_x = 0 \) in Eq. (4.58), we obtain the equation of a straight line, which represents the **neutral axis** of the section:

\[
\frac{M_y}{I_y} y - \frac{M_z}{I_z} z = \frac{P}{A}
\]
EXAMPLE 4.09

A vertical 4.80-kN load is applied as shown on a wooden post of rectangular cross section, 80 by 120 mm (Fig. 4.65). (a) Determine the stress at points A, B, C, and D. (b) Locate the neutral axis of the cross section.

Fig. 4.65

(a) Stresses. The given eccentric load is replaced by an equivalent system consisting of a centric load \( \mathbf{P} \) and two couples \( \mathbf{M}_x \) and \( \mathbf{M}_z \) represented by vectors directed along the principal centroidal axes of the section (Fig. 4.66). We have

\[
\begin{align*}
M_x &= (4.80 \text{ kN})(40 \text{ mm}) = 192 \text{ N} \cdot \text{m} \\
M_z &= (4.80 \text{ kN})(60 \text{ mm} - 35 \text{ mm}) = 120 \text{ N} \cdot \text{m}
\end{align*}
\]

We also compute the area and the centroidal moments of inertia of the cross section:

\[
\begin{align*}
A &= (0.080 \text{ m})(0.120 \text{ m}) = 9.60 \times 10^{-3} \text{ m}^2 \\
I_x &= \frac{1}{12}(0.120 \text{ m})(0.080 \text{ m})^3 = 5.12 \times 10^{-6} \text{ m}^4 \\
I_z &= \frac{1}{12}(0.080 \text{ m})(0.120 \text{ m})^3 = 11.52 \times 10^{-6} \text{ m}^4
\end{align*}
\]

The stress \( \sigma_0 \) due to the centric load \( \mathbf{P} \) is negative and uniform across the section. We have

\[
\sigma_0 = \frac{P}{A} = \frac{-4.80 \text{ kN}}{9.60 \times 10^{-3} \text{ m}^2} = -0.5 \text{ MPa}
\]

The stresses due to the bending couples \( \mathbf{M}_x \) and \( \mathbf{M}_z \) are linearly distributed across the section, with maximum values equal, respectively, to

\[
\begin{align*}
\sigma_1 &= \frac{M_x z_{\text{max}}}{I_x} = \frac{(192 \text{ N} \cdot \text{m})(40 \text{ mm})}{5.12 \times 10^{-6} \text{ m}^4} = 1.5 \text{ MPa} \\
\sigma_2 &= \frac{M_z z_{\text{max}}}{I_z} = \frac{(120 \text{ N} \cdot \text{m})(60 \text{ mm})}{11.52 \times 10^{-6} \text{ m}^4} = 0.625 \text{ MPa}
\end{align*}
\]

The stresses at the corners of the section are

\[
\sigma_y = \sigma_0 \pm \sigma_1 \pm \sigma_2
\]

where the signs must be determined from Fig. 4.66. Noting that the stresses due to \( \mathbf{M}_x \) are positive at \( C \) and \( D \), and negative at \( A \) and \( B \), and
that the stresses due to $M_z$ are positive at $B$ and $C$, and negative at $A$ and $D$, we obtain

$$
\sigma_A = -0.5 - 1.5 - 0.625 = -2.625 \text{ MPa}
$$

$$
\sigma_B = -0.5 - 1.5 + 0.625 = -1.375 \text{ MPa}
$$

$$
\sigma_C = -0.5 + 1.5 + 0.625 = +1.625 \text{ MPa}
$$

$$
\sigma_D = -0.5 + 1.5 - 0.625 = +0.375 \text{ MPa}
$$

![Figure 4.67](image)

**Fig. 4.67**

(b) Neutral Axis. We note that the stress will be zero at a point $G$ between $B$ and $C$, and at a point $H$ between $D$ and $A$ (Fig. 4.67). Since the stress distribution is linear, we write

$$
BG = \frac{1.375}{1.625 + 1.375} \times 80 \text{ mm} = 36.7 \text{ mm}
$$

$$
HA = \frac{2.625}{2.625 + 0.375} \times 80 \text{ mm} = 70 \text{ mm}
$$

The neutral axis can be drawn through points $G$ and $H$ (Fig. 4.68).

![Figure 4.68](image)

**Fig. 4.68**

The distribution of the stresses across the section is shown in Fig. 4.69.

![Figure 4.69](image)

**Fig. 4.69**
SAMPLE PROBLEM 4.9

A horizontal load \( P \) is applied as shown to a short section of an \( S10 \times 25.4 \) rolled-steel member. Knowing that the compressive stress in the member is not to exceed 12 ksi, determine the largest permissible load \( P \).

SOLUTION

Properties of Cross Section. The following data are taken from Appendix C.

\[ \text{Area: } A = 7.46 \text{ in}^2 \]
\[ \text{Section moduli: } S_x = 24.7 \text{ in}^3 \quad S_y = 2.91 \text{ in}^3 \]

Force and Couple at \( C \). We replace \( P \) by an equivalent force-couple system at the centroid \( C \) of the cross section.

\[ M_x = (4.75 \text{ in.}) P \quad M_y = (1.5 \text{ in.}) P \]

Note that the couple vectors \( M_x \) and \( M_y \) are directed along the principal axes of the cross section.

Normal Stresses. The absolute values of the stresses at points \( A, B, D, \) and \( E \) due, respectively, to the centric load \( P \) and to the couples \( M_x \) and \( M_y \) are

\[ \sigma_1 = \frac{P}{A} = \frac{P}{7.46 \text{ in}^2} = 0.1340P \]
\[ \sigma_2 = \frac{M_x}{S_x} = \frac{4.75P}{24.7 \text{ in}^3} = 0.1923P \]
\[ \sigma_3 = \frac{M_y}{S_y} = \frac{1.5P}{2.91 \text{ in}^3} = 0.5155P \]

Superposition. The total stress at each point is found by superposing the stresses due to \( P, M_x, \) and \( M_y \). We determine the sign of each stress by carefully examining the sketch of the force-couple system.

\[ \sigma_A = -\sigma_1 + \sigma_2 + \sigma_3 = -0.1340P + 0.1923P + 0.5155P = +0.574P \]
\[ \sigma_B = -\sigma_1 + \sigma_2 - \sigma_3 = -0.1340P + 0.1923P - 0.5155P = -0.457P \]
\[ \sigma_D = -\sigma_1 - \sigma_2 + \sigma_3 = -0.1340P - 0.1923P + 0.5155P = +0.189P \]
\[ \sigma_E = -\sigma_1 - \sigma_2 - \sigma_3 = -0.1340P - 0.1923P - 0.5155P = -0.842P \]

Largest Permissible Load. The maximum compressive stress occurs at point \( E \). Recalling that \( \sigma_{\text{all}} = -12 \text{ ksi} \), we write

\[ \sigma_{\text{all}} = \sigma_E = -12 \text{ ksi} = -0.842P \quad P = 14.3 \text{ kips} \]
**SAMPLE PROBLEM 4.10**

A couple of magnitude \( M_0 = 1.5 \text{ kN} \cdot \text{m} \) acting in a vertical plane is applied to a beam having the Z-shaped cross section shown. Determine (a) the stress at point \( A \), (b) the angle that the neutral axis forms with the horizontal plane. The moments and product of inertia of the section with respect to the \( y \) and \( z \) axes have been computed and are as follows:

\[
I_y = 3.25 \times 10^{-6} \text{ m}^4 \quad I_z = 4.18 \times 10^{-6} \text{ m}^4 \quad I_{yz} = 2.87 \times 10^{-6} \text{ m}^4
\]

**SOLUTION**

**Principal Axes.** We draw Mohr’s circle and determine the orientation of the principal axes and the corresponding principal moments of inertia.

\[
\tan 2\theta_m = \frac{FZ}{EF} = \frac{2.87}{0.465} \quad 2\theta_m = 80.8^\circ \quad \theta_m = 40.4^\circ
\]

\[
R^2 = (EF)^2 + (FZ)^2 = (0.465)^2 + (2.87)^2 \quad R = 2.91 \times 10^{-6} \text{ m}^4
\]

\[
I_u = I_{\text{min}} = OU = I_{\text{ave}} - R = 3.72 - 2.91 = 0.810 \times 10^{-6} \text{ m}^4
\]

\[
I_v = I_{\text{max}} = OV = I_{\text{ave}} + R = 3.72 + 2.91 = 6.63 \times 10^{-6} \text{ m}^4
\]

**Loading.** The applied couple \( M_0 \) is resolved into components parallel to the principal axes.

\[
M_u = M_0 \sin \theta_m = 1500 \sin 40.4^\circ = 972 \text{ N} \cdot \text{m}
\]

\[
M_v = M_0 \cos \theta_m = 1500 \cos 40.4^\circ = 1142 \text{ N} \cdot \text{m}
\]

**a. Stress at \( A \).** The perpendicular distances from each principal axis to point \( A \) are

\[
u_A = y_A \cos \theta_m + z_A \sin \theta_m = 50 \cos 40.4^\circ + 74 \sin 40.4^\circ = 86.0 \text{ mm}
\]

\[
u_A = -y_A \sin \theta_m + z_A \cos \theta_m = -50 \sin 40.4^\circ + 74 \cos 40.4^\circ = 23.9 \text{ mm}
\]

Considering separately the bending about each principal axis, we note that \( M_u \) produces a tensile stress at point \( A \) while \( M_v \) produces a compressive stress at the same point.

\[
\sigma_A = \frac{M_u v_A}{I_u} = \frac{M_u y_A}{I_u} = \frac{(972 \text{ N} \cdot \text{m})(0.0239 \text{ m})}{0.810 \times 10^{-6} \text{ m}^4} = \frac{(1142 \text{ N} \cdot \text{m})(0.0860 \text{ m})}{6.63 \times 10^{-6} \text{ m}^4}
\]

\[
= +28.68 \text{ MPa} \quad \sigma_A = +13.87 \text{ MPa}
\]

**b. Neutral Axis.** Using Eq. (4.57), we find the angle \( \phi \) that the neutral axis forms with the \( v \) axis.

\[
\tan \phi = \frac{I_v}{I_u} \tan \theta_m = \frac{6.63}{0.810} \tan 40.4^\circ \quad \phi = 81.8^\circ
\]

The angle \( \beta \) formed by the neutral axis and the horizontal is

\[
\beta = \phi - \theta_m = 81.8^\circ - 40.4^\circ = 41.4^\circ
\]

4.127 through 4.134 The couple \( M \) is applied to a beam of the cross section shown in a plane forming an angle \( \beta \) with the vertical. Determine the stress at (a) point \( A \), (b) point \( B \), (c) point \( D \).

![Fig. P4.127](image1)

![Fig. P4.128](image2)

![Fig. P4.129](image3)

![Fig. P4.130](image4)

![Fig. P4.131](image5)

![Fig. P4.132](image6)

![Fig. P4.133](image7)

![Fig. P4.134](image8)
4.135 through 4.140 The couple \( M \) acts in a vertical plane and is applied to a beam oriented as shown. Determine \((a)\) the angle that the neutral axis forms with the horizontal, \((b)\) the maximum tensile stress in the beam.
*4.141 through *4.143 The couple \( \mathbf{M} \) acts in a vertical plane and is applied to a beam oriented as shown. Determine the stress at point \( A \).

4.144 The tube shown has a uniform wall thickness of 12 mm. For the loading given, determine (a) the stress at points \( A \) and \( B \), (b) the point where the neutral axis intersects line \( ABD \).

4.145 Solve Prob. 4.144, assuming that the 28-kN force at point \( E \) is removed.

4.146 A rigid circular plate of 125-mm radius is attached to a solid 150 × 200-mm rectangular post, with the center of the plate directly above the center of the post. If a 4-kN force \( \mathbf{P} \) is applied at \( E \) with \( \theta = 30^\circ \), determine (a) the stress at point \( A \), (b) the stress at point \( B \), (c) the point where the neutral axis intersects line \( ABD \).

4.147 In Prob. 4.146, determine (a) the value of \( \theta \) for which the stress at \( D \) reaches its largest value, (b) the corresponding values of the stress at \( A \), \( B \), \( C \), and \( D \).
4.148 Knowing that $P = 90$ kips, determine the largest distance $a$ for which the maximum compressive stress does not exceed 18 ksi.

4.149 Knowing that $a = 1.25$ in., determine the largest value of $P$ that can be applied without exceeding either of the following allowable stresses:

$$\sigma_{\text{ten}} = 10 \text{ ksi} \quad \sigma_{\text{comp}} = 18 \text{ ksi}$$

4.150 The $Z$ section shown is subjected to a couple $M_0$ acting in a vertical plane. Determine the largest permissible value of the moment $M_0$ of the couple if the maximum stress is not to exceed 80 MPa. Given: $I_{\text{max}} = 2.28 \times 10^{-4}$ m$^4$, $I_{\text{min}} = 0.23 \times 10^{-4}$ m$^4$, principal axes $25.7^\circ \angle \sigma$ and $64.3^\circ \angle \tau$.

4.151 Solve Prob. 4.150, assuming that the couple $M_0$ acts in a horizontal plane.

4.152 A beam having the cross section shown is subjected to a couple $M_0$ that acts in a vertical plane. Determine the largest permissible value of the moment $M_0$ of the couple if the maximum stress in the beam is not to exceed 12 ksi. Given: $I_y = I_z = 11.3$ in$^4$, $A = 4.75$ in$^2$, $k_{\text{min}} = 0.983$ in. (Hint: By reason of symmetry, the principal axes form an angle of $45^\circ$ with the coordinate axes. Use the relations $I_{\text{min}} = Ak_{\text{min}}^2$ and $I_{\text{min}} + I_{\text{max}} = I_y + I_z$.)

4.153 Solve Prob. 4.152, assuming that the couple $M_0$ acts in a horizontal plane.

4.154 An extruded aluminum member having the cross section shown is subjected to a couple acting in a vertical plane. Determine the largest permissible value of the moment $M_0$ of the couple if the maximum stress is not to exceed 12 ksi. Given: $I_{\text{max}} = 0.957$ in$^4$, $I_{\text{min}} = 0.427$ in$^4$, principal axes $29.4^\circ \angle \sigma$ and $60.6^\circ \angle \tau$.

4.155 A couple $M_0$ acting in a vertical plane is applied to a W12 × 16 rolled-steel beam, whose web forms an angle $\theta$ with the vertical. Denoting by $\sigma_0$ the maximum stress in the beam when $\theta = 0$, determine the angle of inclination $\theta$ of the beam for which the maximum stress is $2\sigma_0$. 

---

**Fig. P4.148 and P4.149**

**Fig. P4.149**

**Fig. P4.152**

**Fig. P4.152**

**Fig. P4.154**

**Fig. P4.155**
4.156 Show that, if a solid rectangular beam is bent by a couple applied in a plane containing one diagonal of a rectangular cross section, the neutral axis will lie along the other diagonal.

4.157 A beam of unsymmetric cross section is subjected to a couple \( \mathbf{M}_0 \) acting in the horizontal plane \( \text{xy} \). Show that the stress at point \( A \), of coordinates \( y \) and \( z \), is

\[
\sigma_A = \frac{zI_y - yI_z}{I_y I_z - I_y I_z} M_y
\]

where \( I_y \), \( I_z \), and \( I_yz \) denote the moments and product of inertia of the cross section with respect to the coordinate axes, and \( M_y \) the moment of the couple.

4.158 A beam of unsymmetric cross section is subjected to a couple \( \mathbf{M}_0 \) acting in the vertical plane \( \text{xy} \). Show that the stress at point \( A \), of coordinates \( y \) and \( z \), is

\[
\sigma_A = -\frac{yI_y - zI_z}{I_y I_z - I_y I_z} M_z
\]

where \( I_y \), \( I_z \), and \( I_yz \) denote the moments and product of inertia of the cross section with respect to the coordinate axes, and \( M_z \) the moment of the couple.

4.159 (a) Show that, if a vertical force \( P \) is applied at point \( A \) of the section shown, the equation of the neutral axis \( BD \) is

\[
\left( \frac{x_A}{r_x^2} \right) x + \left( \frac{z_A}{r_z^2} \right) z = -1
\]

where \( r_x \) and \( r_z \) denote the radius of gyration of the cross section with respect to the \( x \) axis and the \( z \) axis, respectively. (b) Further show that, if a vertical force \( Q \) is applied at any point located on line \( BD \), the stress at point \( A \) will be zero.

4.160 (a) Show that the stress at corner \( A \) of the prismatic member shown in Fig. P.4.160a will be zero if the vertical force \( P \) is applied at a point located on the line

\[
\frac{x}{b/6} + \frac{z}{h/6} = 1
\]

(b) Further show that, if no tensile stress is to occur in the member, the force \( P \) must be applied at a point located within the area bounded by the line found in part \( a \) and three similar lines corresponding to the condition of zero stress at \( B \), \( C \), and \( D \), respectively. This area, shown in Fig. P.4.160b, is known as the kern of the cross section.
*4.15 BENDING OF CURVED MEMBERS

Our analysis of stresses due to bending has been restricted so far to straight members. In this section we will consider the stresses caused by the application of equal and opposite couples to members that are initially curved. Our discussion will be limited to curved members of uniform cross section possessing a plane of symmetry in which the bending couples are applied, and it will be assumed that all stresses remain below the proportional limit.

If the initial curvature of the member is small, i.e., if its radius of curvature is large compared to the depth of its cross section, a good approximation can be obtained for the distribution of stresses by assuming the member to be straight and using the formulas derived in Secs. 4.3 and 4.4.† However, when the radius of curvature and the dimensions of the cross section of the member are of the same order of magnitude, we must use a different method of analysis, which was first introduced by the German engineer E. Winkler (1835–1888).

Consider the curved member of uniform cross section shown in Fig. 4.70. Its transverse section is symmetric with respect to the $y$ axis (Fig. 4.70b) and, in its unstressed state, its upper and lower surfaces intersect the vertical $xy$ plane along arcs of circle $AB$ and $FG$ centered at $C$ (Fig. 4.70a). We now apply two equal and opposite couples $M$ and $M'$ in the plane of symmetry of the member (Fig. 4.70c). A reasoning similar to that of Sec. 4.3 would show that any transverse plane section containing $C$ will remain plane, and that the various arcs of circle indicated in Fig. 4.70a will be transformed into circular and concentric arcs with a center $C'$ different from $C$. More specifically, if the couples $M$ and $M'$ are directed as shown, the curvature of the various arcs of circle will increase; that is $A'C' < AC$. We also note that the couples $M$ and $M'$ will cause the length of the upper surface

†See Prob. 4.166.
of the member to decrease ($A'B' < AB$) and the length of the lower surface to increase ($F'G' > FG$). We conclude that a neutral surface must exist in the member, the length of which remains constant. The intersection of the neutral surface with the $xy$ plane has been represented in Fig. 4.70a by the arc $DE$ of radius $R$, and in Fig. 4.70c by the arc $D'E'$ of radius $R'$. Denoting by $\theta$ and $\theta'$ the central angles corresponding respectively to $DE$ and $D'E'$, we express the fact that the length of the neutral surface remains constant by writing

$$R\theta = R'\theta' \tag{4.59}$$

Considering now the arc of circle $JK$ located at a distance $y$ above the neutral surface, and denoting respectively by $r$ and $r'$ the radius of this arc before and after the bending couples have been applied, we express the deformation of $JK$ as

$$\delta = r'\theta' - r\theta \tag{4.60}$$

Observing from Fig. 4.70 that

$$r = R - y \quad r' = R' - y \tag{4.61}$$

and substituting these expressions into Eq. (4.60), we write

$$\delta = (R' - y)\theta' - (R - y)\theta$$

or, recalling Eq. (4.59) and setting $\theta' - \theta = \Delta\theta$,

$$\delta = -y \Delta\theta \tag{4.62}$$

The normal strain $\epsilon_x$ in the elements of $JK$ is obtained by dividing the deformation $\delta$ by the original length $r\theta$ of arc $JK$. We write

$$\epsilon_x = \frac{\delta}{r\theta} = \frac{-y \Delta\theta}{r\theta}$$

or, recalling the first of the relations (4.61),

$$\epsilon_x = \frac{\Delta\theta}{\theta} \frac{y}{R - y} \tag{4.63}$$

The relation obtained shows that, while each transverse section remains plane, the normal strain $\epsilon_x$ does not vary linearly with the distance $y$ from the neutral surface.

The normal stress $\sigma_x$ can now be obtained from Hooke’s law, $\sigma_x = E\epsilon_x$, by substituting for $\epsilon_x$ from Eq. (4.63). We have

$$\sigma_x = \frac{E \Delta\theta}{\theta} \frac{y}{R - y} \tag{4.64}$$

or, alternatively, recalling the first of Eqs. (4.61),

$$\sigma_x = \frac{E \Delta\theta}{\theta} \frac{R - r}{r} \tag{4.65}$$

Equation (4.64) shows that, like $\epsilon_x$, the normal stress $\sigma_x$ does not vary linearly with the distance $y$ from the neutral surface. Plotting $\sigma_x$ versus $y$, we obtain an arc of hyperbola (Fig. 4.71).

In order to determine the location of the neutral surface in the member and the value of the coefficient $E \Delta\theta/\theta$ used in Eqs. (4.64)
and (4.65), we now recall that the elementary forces acting on any transverse section must be statically equivalent to the bending couple $M$. Expressing, as we did in Sec. 4.2 for a straight member, that the sum of the elementary forces acting on the section must be zero, and that the sum of their moments about the transverse $z$ axis must be equal to the bending moment $M$, we write the equations

$$\int \sigma_x \, dA = 0$$  \hspace{1cm} (4.1)

and

$$\int (-y\sigma_x \, dA) = M$$  \hspace{1cm} (4.3)

Substituting for $\sigma_x$ from (4.65) into Eq. (4.1), we write

$$\int \frac{E \Delta \theta}{\theta} \frac{R - r}{r} \, dA = 0$$

$$\int \frac{R - r}{r} \, dA = 0$$

$$R \int \frac{dA}{r} - \int dA = 0$$

from which it follows that the distance $R$ from the center of curvature $C$ to the neutral surface is defined by the relation

$$R = \frac{A}{\int \frac{dA}{r}}$$  \hspace{1cm} (4.66)

We note that the value obtained for $R$ is not equal to the distance $\bar{r}$ from $C$ to the centroid of the cross section, since $\bar{r}$ is defined by a different relation, namely,

$$\bar{r} = \frac{1}{A} \int r \, dA$$  \hspace{1cm} (4.67)

We thus conclude that, in a curved member, the neutral axis of a transverse section does not pass through the centroid of that section (Fig. 4.72).† Expressions for the radius $R$ of the neutral surface will be derived for some specific cross-sectional shapes in Example 4.10 and in Probs. 4.188 through 4.190. For convenience, these expressions are shown in Fig. 4.73.

Substituting now for $\sigma_x$ from (4.65) into Eq. (4.3), we write

$$\int \frac{E \Delta \theta}{\theta} \frac{R - r}{r} y \, dA = M$$

†However, an interesting property of the neutral surface can be noted if we write Eq. (4.66) in the alternative form

$$\frac{1}{R} = \frac{1}{A} \int \frac{1}{r} \, dA$$  \hspace{1cm} (4.66')

Equation (4.66') shows that, if the member is divided into a large number of fibers of cross-sectional area $dA$, the curvature $1/R$ of the neutral surface will be equal to the average value of the curvature $1/r$ of the various fibers.
Expanding the square in the integrand, we obtain after reductions

\[ E \frac{\Delta \theta}{\theta} \left[ \frac{R^2}{r} \int \frac{dA}{r} - 2RA + \int r \, dA \right] = M \]

Recalling Eqs. (4.66) and (4.67), we note that the first term in the brackets is equal to RA, while the last term is equal to \( \bar{r}A \). We have, therefore,

\[ E \frac{\Delta \theta}{\theta} (RA - 2RA + \bar{r}A) = M \]

and, solving for \( E \Delta \theta / \theta \),

\[ E \frac{\Delta \theta}{\theta} = \frac{M}{A(\bar{r} - R)} \quad (4.68) \]

Referring to Fig. 4.70, we note that \( \Delta \theta > 0 \) for \( M > 0 \). It follows that \( \bar{r} - R > 0 \), or \( R < \bar{r} \), regardless of the shape of the section. Thus, the neutral axis of a transverse section is always located between the centroid of the section and the center of curvature of the member (Fig. 4.72). Setting \( \bar{r} - R = e \), we write Eq. (4.68) in the form

\[ E \frac{\Delta \theta}{\theta} = \frac{M}{Ae} \quad (4.69) \]

Substituting now for \( E \Delta \theta / \theta \) from (4.69) into Eqs. (4.64) and (4.65), we obtain the following alternative expressions for the normal stress \( \sigma_x \) in a curved beam:

\[ \sigma_x = -\frac{My}{Ae(R - y)} \quad (4.70) \]

and

\[ \sigma_x = \frac{M(r - R)}{Aer} \quad (4.71) \]
We should note that the parameter $e$ in the previous equations is a small quantity obtained by subtracting two lengths of comparable size, $R$ and $\bar{r}$. In order to determine $\sigma_x$ with a reasonable degree of accuracy, it is therefore necessary to compute $R$ and $\bar{r}$ very accurately, particularly when both of these quantities are large, i.e., when the curvature of the member is small. However, as we indicated earlier, it is possible in such a case to obtain a good approximation for $\sigma_x$ by using the formula $\sigma_x = -My/I$ developed for straight members.

Let us now determine the change in curvature of the neutral surface caused by the bending moment $M$. Solving Eq. (4.59) for the curvature $1/R'$ of the neutral surface in the deformed member, we write

$$\frac{1}{R'} = \frac{1}{R} \frac{\theta'}{\theta}$$

or, setting $\theta' = \theta + \Delta\theta$ and recalling Eq. (4.69),

$$\frac{1}{R'} = \frac{1}{R} \left(1 + \frac{\Delta\theta}{\theta}\right) = \frac{1}{R} \left(1 + \frac{M}{EA\epsilon}\right)$$

from which it follows that the change in curvature of the neutral surface is

$$\frac{1}{R'} - \frac{1}{R} = \frac{M}{EA\epsilon}$$

(4.72)

**EXAMPLE 4.10**

A curved rectangular bar has a mean radius $\bar{r} = 6$ in. and a cross section with $b = 2.5$ in. and $h = 1.5$ in. (Fig. 4.74). Determine the distance $e$ between the centroid and the neutral axis of the cross section.

We first derive the expression for the radius $R$ of the neutral surface. Denoting by $r_1$ and $r_2$, respectively, the inner and outer radius of the bar (Fig. 4.75), we use Eq. (4.66) and write

$$R = \frac{A}{\int_{r_1}^{r_2} \frac{dA}{r}} = \frac{bh}{\int_{r_1}^{r_2} \frac{b}{r} \, dr} = \frac{h}{\int_{r_1}^{r_2} \frac{dr}{r}}$$

$$R = \frac{h}{\ln\frac{r_2}{r_1}}$$

(4.73)
For the given data, we have
\[ r_1 = \frac{1}{2}h = 6 - 0.75 = 5.25 \text{ in.} \]
\[ r_2 = \frac{1}{2}h = 6 + 0.75 = 6.75 \text{ in.} \]
Substituting for \( h, r_1, \) and \( r_2 \) into Eq. (4.73), we have
\[ R = \frac{h}{\ln \frac{r_2}{r_1}} = \frac{1.5 \text{ in.}}{\ln \frac{6.75}{5.25}} = 5.9686 \text{ in.} \]
The distance between the centroid and the neutral axis of the cross section (Fig. 4.76) is thus
\[ e = \bar{r} - R = 6 - 5.9686 = 0.0314 \text{ in.} \]
We note that it was necessary to calculate \( R \) with five significant figures in order to obtain \( e \) with the usual degree of accuracy.

For the bar of Example 4.10, determine the largest tensile and compressive stresses, knowing that the bending moment in the bar is \( M = 8 \text{ kip} \cdot \text{in.} \).

We use Eq. (4.71) with the given data,
\[ M = 8 \text{ kip} \cdot \text{in.} \quad A = bh = (2.5 \text{ in.})(1.5 \text{ in.}) = 3.75 \text{ in}^2 \]
and the values obtained in Example 4.10 for \( R \) and \( e \),
\[ R = 5.969 \quad e = 0.0314 \text{ in.} \]
Making first \( r = r_2 = 6.75 \text{ in.} \) in Eq. (4.71), we write
\[ \sigma_{\text{max}} = \frac{M(r_2 - R)}{Ae r_2} = \frac{(8 \text{ kip} \cdot \text{in.})(6.75 \text{ in.} - 5.969 \text{ in.})}{(3.75 \text{ in}^2)(0.0314 \text{ in.})(6.75 \text{ in.})} = 7.86 \text{ ksi} \]
Making now \( r = r_1 = 5.25 \text{ in.} \) in Eq. (4.71), we have
\[ \sigma_{\text{min}} = \frac{M(r_1 - R)}{Ae r_1} = \frac{(8 \text{ kip} \cdot \text{in.})(5.25 \text{ in.} - 5.969 \text{ in.})}{(3.75 \text{ in}^2)(0.0314 \text{ in.})(5.25 \text{ in.})} = -9.30 \text{ ksi} \]
**Remark.** Let us compare the values obtained for \( \sigma_{\text{max}} \) and \( \sigma_{\text{min}} \) with the result we would get for a straight bar. Using Eq. (4.15) of Sec. 4.4, we write
\[ \sigma_{\text{max}, \text{min}} = \pm \frac{Mc}{I} = \pm \frac{(8 \text{ kip} \cdot \text{in.})(0.75 \text{ in.})}{\frac{1}{12}(2.5 \text{ in.})(1.5 \text{ in.})^3} = \pm 8.53 \text{ ksi} \]
SAMPLE PROBLEM 4.11

A machine component has a T-shaped cross section and is loaded as shown. Knowing that the allowable compressive stress is 50 MPa, determine the largest force \( P \) that can be applied to the component.

SOLUTION

**Centroid of the Cross Section.** We locate the centroid \( D \) of the cross section:

\[
\sum A_i \bar{r}_i = \sum r_i A_i
\]

<table>
<thead>
<tr>
<th>( A_i ), mm(^2 )</th>
<th>( r_i ), mm</th>
<th>( r_i A_i ), mm(^3 )</th>
<th>( \bar{r} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20</td>
<td>1800</td>
<td>30</td>
</tr>
<tr>
<td>2</td>
<td>40</td>
<td>800</td>
<td>70</td>
</tr>
</tbody>
</table>

\[
\bar{r} = \frac{64 \times 10^3 \bar{r}(2400) = 120 \times 10^3}{120 \times 10^3} = 50 \text{ mm} = 0.050 \text{ m}
\]

**Force and Couple at \( D \).** The internal forces in section \( a\)-\( a \) are equivalent to a force \( P \) acting at \( D \) and a couple \( M \) of moment

\[
M = P(50 \text{ mm} - 60 \text{ mm}) = (0.110 \text{ m})P
\]

**Superposition.** The centric force \( P \) causes a uniform compressive stress on section \( a\)-\( a \). The bending couple \( M \) causes a varying stress distribution [Eq. (4.71)]. We note that the couple \( M \) tends to increase the curvature of the member and is therefore positive (cf. Fig. 4.70). The total stress at a point of section \( a\)-\( a \) located at distance \( r \) from the center of curvature \( C \) is

\[
\sigma = -\frac{P}{A} + \frac{M(r - R)}{Aer}
\]

**Radius of Neutral Surface.** We now determine the radius \( R \) of the neutral surface by using Eq. (4.66).

\[
R = A \int \frac{dA}{r} = \frac{2400 \text{ mm}^2}{80 \ln \frac{50}{30} + 20 \ln \frac{90}{50}} = 45.61 \text{ mm}
\]

\[
= 0.04561 \text{ m}
\]

We also compute: \( e = \bar{r} - R = 0.05000 \text{ m} - 0.04561 \text{ m} = 0.00439 \text{ m} \)

**Allowable Load.** We observe that the largest compressive stress will occur at point \( A \) where \( r = 0.030 \text{ m} \). Recalling that \( \sigma_{\text{all}} = 50 \text{ MPa} \) and using Eq. (1), we write

\[
-50 \times 10^6 \text{ Pa} = - \frac{P}{2.4 \times 10^{-3} \text{ m}^2} + \frac{(0.110 \text{ P})(0.030 \text{ m} - 0.04561 \text{ m})}{(2.4 \times 10^{-3} \text{ m}^2)(0.00439 \text{ m})(0.030 \text{ m})}
\]

\[
-50 \times 10^6 = 417P - 5432P
\]

\[
P = 8.55 \text{ kN}
\]
4.161 For the machine component and loading shown, determine the stress at point A when (a) $h = 2$ in., (b) $h = 2.6$ in.

4.162 For the machine component and loading shown, determine the stress at points A and B when $h = 2.5$ in.

4.163 The curved portion of the bar shown has an inner radius of 20 mm. Knowing that the allowable stress in the bar is 150 MPa, determine the largest permissible distance $a$ from the line of action of the 3-kN force to the vertical plane containing the center of curvature of the bar.

4.164 The curved portion of the bar shown has an inner radius of 20 mm. Knowing that the line of action of the 3-kN force is located at a distance $a = 60$ mm from the vertical plane containing the center of curvature of the bar, determine the largest compressive stress in the bar.

4.165 The curved bar shown has a cross section of $40 \times 60$ mm and an inner radius $r_1 = 15$ mm. For the loading shown determine the largest tensile and compressive stresses.

4.166 For the curved bar and loading shown, determine the percent error introduced in the computation of the maximum stress by assuming that the bar is straight. Consider the case when (a) $r_1 = 20$ mm, (b) $r_1 = 200$ mm, (c) $r_1 = 2$ m.

4.167 The curved bar shown has a cross section of $30 \times 30$ mm. Knowing that $a = 60$ mm, determine the stress at (a) point A, (b) point B.

4.168 The curved bar shown has a cross section of $30 \times 30$ mm. Knowing that the allowable compressive stress is 175 MPa, determine the largest allowable distance $a$. 
Steel links having the cross section shown are available with different central angles \( \beta \). Knowing that the allowable stress is 12 ksi, determine the largest force \( P \) that can be applied to a link for which \( \beta = 90^\circ \).

![Fig. P4.169](image)

Solve Prob. 4.169, assuming that \( \beta = 60^\circ \).

A machine component has a T-shaped cross section that is oriented as shown. Knowing that \( M = 2.5 \text{ kN} \cdot \text{m} \), determine the stress at (a) point \( A \), (b) point \( B \).

Assuming that the couple shown is replaced by a vertical 10-kN force attached at point \( D \) and acting downward, determine the stress at (a) point \( A \), (b) point \( B \).

Three plates are welded together to form the curved beam shown. For the given loading, determine the distance \( e \) between the neutral axis and the centroid of the cross section.

Three plates are welded together to form the curved beam shown. For \( M = 8 \text{ kip} \cdot \text{in} \), determine the stress at (a) point \( A \), (b) point \( B \), (c) the centroid of the cross section.

The split ring shown has an inner radius \( r_1 = 20 \text{ mm} \) and a circular cross section of diameter \( d = 32 \text{ mm} \). For the loading shown, determine the stress at (a) point \( A \), (b) point \( B \).

The split ring shown has an inner radius \( r_1 = 16 \text{ mm} \) and a circular cross section of diameter \( d = 32 \text{ mm} \). For the loading shown, determine the stress at (a) point \( A \), (b) point \( B \).
4.177 The curved bar shown has a circular cross section of 32-mm diameter. Determine the largest couple \( M \) that can be applied to the bar about a horizontal axis if the maximum stress is not to exceed 60 MPa.

![Fig. P4.177](image)

4.178 The bar shown has a circular cross section of 0.6 in.-diameter. Knowing that \( a = 1.2 \) in., determine the stress at (a) point A, (b) point B.

4.179 The bar shown has a circular cross section of 0.6-in. diameter. Knowing that the allowable stress is 8 ksi, determine the largest permissible distance \( a \) from the line of action of the 50-lb forces to the plane containing the center of curvature of the bar.

4.180 Knowing that \( P = 10 \) kN, determine the stress at (a) point A, (b) point B.

4.181 and 4.182 Knowing that \( M = 5 \) kip \( \cdot \) in., determine the stress at (a) point A, (b) point B.

4.183 For the curved beam and loading shown, determine the stress at (a) point A, (b) point B.

4.184 For the crane hook shown, determine the largest tensile stress in section \( a-a \).
4.185 Knowing that the machine component shown has a trapezoidal cross section with \(a = 3.5\) in. and \(b = 2.5\) in., determine the stress at (a) point A, (b) point B.

4.186 Knowing that the machine component shown has a trapezoidal cross section with \(a = 2.5\) in. and \(b = 3.5\) in., determine the stress at (a) point A, (b) point B.

4.187 Show that if the cross section of a curved beam consists of two or more rectangles, the radius \(R\) of the neutral surface can be expressed as

\[
R = \frac{A}{\ln \left( \frac{rocation2}{rocation1} \right) \left( \frac{rocation3}{rocation2} \right)^{b2} \left( \frac{rocation4}{rocation3} \right)^{b3}}
\]

where \(A\) is the total area of the cross section.

4.188 through 4.190 Using Eq. (4.66), derive the expression for \(R\) given in Fig. 4.73 for

*4.188* A circular cross section.

*4.189* A trapezoidal cross section.

*4.190* A triangular cross section.

*4.191* For a curved bar of rectangular cross section subjected to a bending couple \(M\), show that the radial stress at the neutral surface is

\[
\sigma_r = \frac{M}{Ae} \left( 1 - \frac{rocation1}{R} - \ln \frac{R}{rocation1} \right)
\]

and compute the value of \(\sigma_r\) for the curved bar of Examples 4.10 and 4.11.

(Hint: consider the free-body diagram of the portion of the beam located above the neutral surface.)
This chapter was devoted to the analysis of members in pure bending. That is, we considered the stresses and deformation in members subjected to equal and opposite couples \( M \) and \( M' \) acting in the same longitudinal plane (Fig. 4.77).

We first studied members possessing a plane of symmetry and subjected to couples acting in that plane. Considering possible deformations of the member, we proved that transverse sections remain plane as a member is deformed [Sec. 4.3]. We then noted that a member in pure bending has a neutral surface along which normal strains and stresses are zero and that the longitudinal normal strain \( \epsilon_x \) varies linearly with the distance \( y \) from the neutral surface:

\[
\epsilon_x = -\frac{y}{\rho} \quad (4.8)
\]

where \( \rho \) is the radius of curvature of the neutral surface (Fig. 4.78). The intersection of the neutral surface with a transverse section is known as the neutral axis of the section.

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For members made of a material that follows Hooke’s law [Sec. 4.4], we found that the normal stress \( \sigma_x \) varies linearly with the distance from the neutral axis (Fig. 4.79). Denoting by \( \sigma_m \) the maximum stress we wrote:

\[
\sigma_x = -\frac{y}{c} \sigma_m \quad (4.12)
\]

where \( c \) is the largest distance from the neutral axis to a point in the section.

By setting the sum of the elementary forces, \( \sigma_x \, dA \), equal to zero, we proved that the neutral axis passes through the centroid of the cross section of a member in pure bending. Then by setting the sum of the moments of the elementary forces equal to the bending moment, we derived the elastic flexure formula for the maximum normal stress:

\[
\sigma_m = \frac{Mc}{I} \quad (4.15)
\]

where \( I \) is the moment of inertia of the cross section with respect to the neutral axis. We also obtained the normal stress at any distance \( y \) from the neutral axis:

\[
\sigma_x = -\frac{My}{I} \quad (4.16)
\]
Noting that \( I \) and \( c \) depend only on the geometry of the cross section, we introduced the *elastic section modulus*

\[
S = \frac{I}{c}
\]

(4.17)

and then used the section modulus to write an alternative expression for the maximum normal stress:

\[
\sigma_m = \frac{M}{S}
\]

(4.18)

**Elastic section modulus**

**Curvature of member** Recalling that the curvature of a member is the reciprocal of its radius of curvature, we expressed the *curvature* of the member as

\[
\frac{1}{\rho} = \frac{M}{EI}
\]

(4.21)

**Anticlastic curvature** In Sec. 4.5, we completed our study of the bending of homogeneous members possessing a plane of symmetry by noting that deformations occur in the plane of a transverse cross section and result in *anticlastic curvature* of the members.

**Members made of several materials** Next we considered the bending of members made of several materials with different moduli of elasticity [Sec. 4.6]. While transverse sections remain plane, we found that, in general, the *neutral axis does not pass through the centroid* of the composite cross section (Fig. 4.80). Using the ratio of the moduli of elasticity of the materials, we obtained a *transformed section* corresponding to an equivalent member made entirely of one material. We then used the methods previously developed to determine the stresses in this equivalent homogeneous member (Fig. 4.81) and then again used the ratio of the moduli of elasticity to determine the stresses in the composite beam [Sample Probs. 4.3 and 4.4].

**Stress concentrations** In Sec. 4.7, *stress concentrations* that occur in members in pure bending were discussed and charts giving stress-concentration factors for flat bars with fillets and grooves were presented in Figs. 4.27 and 4.28.
We next investigated members made of materials that do not follow Hooke’s law [Sec. 4.8]. A rectangular beam made of an elasto-plastic material (Fig. 4.82) was analyzed as the magnitude of the bending moment was increased. The maximum elastic moment $M_Y$ occurred when yielding was initiated in the beam (Fig. 4.83). As the bending moment was further increased, plastic zones developed and the size of the elastic core of the member decreased [Sec. 4.9]. Finally the beam became fully plastic and we obtained the maximum or plastic moment $M_p$. In Sec. 4.11, we found that permanent deformations and residual stresses remain in a member after the loads that caused yielding have been removed.

In Sec. 4.12, we studied the stresses in members loaded eccentrically in a plane of symmetry. Our analysis made use of methods developed earlier. We replaced the eccentric load by a force-couple system located at the centroid of the cross section (Fig. 4.84) and then superposed stresses due to the centric load and the bending couple (Fig. 4.85):

$$\sigma_z = \frac{P}{A} - \frac{M_y}{I}$$  \hspace{1cm} (4.50)
The bending of members of *unsymmetric cross section* was considered next [Sec. 4.13]. We found that the flexure formula may be used, provided that the couple vector $\mathbf{M}$ is directed along one of the principal centroidal axes of the cross section. When necessary we resolved $\mathbf{M}$ into components along the principal axes and superposed the stresses due to the component couples (Figs. 4.86 and 4.87).

$$\sigma_x = -\frac{M_{y}y}{I_{z}} + \frac{M_{z}z}{I_{y}}$$  \hspace{1cm} (4.55)

For the couple $\mathbf{M}$ shown in Fig. 4.88, we determined the orientation of the neutral axis by writing

$$\tan \phi = \frac{I_{z}}{I_{y}} \tan \theta$$  \hspace{1cm} (4.57)

The general case of *eccentric axial loading* was considered in Sec. 4.14, where we again replaced the load by a force-couple system located at the centroid. We then superposed the stresses due to the centric load and two component couples directed along the principal axes:

$$\sigma_x = \frac{P}{A} - \frac{M_{y}y}{I_{z}} + \frac{M_{z}z}{I_{y}}$$  \hspace{1cm} (4.58)

The chapter concluded with the analysis of stresses in *curved members* (Fig. 4.89). While transverse sections remain plane when the member is subjected to bending, we found that the stresses do *not vary linearly* and the neutral surface does not pass through the centroid of the section. The distance $R$ from the center of curvature of the member to the neutral surface was found to be

$$R = \frac{A}{\int_{r}^{dA}}$$  \hspace{1cm} (4.66)

where $A$ is the area of the cross section. The normal stress at a distance $y$ from the neutral surface was expressed as

$$\sigma_x = -\frac{My}{Ae(R - y)}$$  \hspace{1cm} (4.70)

where $M$ is the bending moment and $e$ the distance from the centroid of the section to the neutral surface.
4.192 Two vertical forces are applied to a beam of the cross section shown. Determine the maximum tensile and compressive stresses in portion BC of the beam.

4.193 Straight rods of 6-mm diameter and 30-m length are stored by coiling the rods inside a drum of 1.25-m inside diameter. Assuming that the yield strength is not exceeded, determine (a) the maximum stress in a coiled rod, (b) the corresponding bending moment in the rod. Use $E = 200 \text{ GPa}$.

4.194 Knowing that for the beam shown the allowable stress is 12 ksi in tension and 16 ksi in compression, determine the largest couple $M$ that can be applied.

4.195 In order to increase corrosion resistance, a 2-mm thick cladding of aluminum has been added to a steel bar as shown. The modulus of elasticity is 200 GPa for steel and 70 GPa for aluminum. For a bending moment of 300 N·m, determine (a) the maximum stress in the steel, (b) the maximum stress in the aluminum, (c) the radius of curvature of the bar.

4.196 A single vertical force $P$ is applied to a short steel post as shown. Gages located at A, B, and C indicate the following strains:

$$\varepsilon_A = -500 \text{ } \mu \quad \varepsilon_B = -1000 \text{ } \mu \quad \varepsilon_C = -200 \text{ } \mu$$

Knowing that $E = 29 \times 10^6 \text{ psi}$, determine (a) the magnitude of $P$, (b) the line of action of $P$, (c) the corresponding strain at the hidden edge of the post, where $x = -2.5 \text{ in.}$ and $z = -1.5 \text{ in.}$.
4.197 For the split ring shown, determine the stress at (a) point A, (b) point B.

4.198 A couple $M$ of moment 8 kN · m acting in a vertical plane is applied to a W200 × 19.3 rolled-steel beam as shown. Determine (a) the angle that the neutral axis forms with the horizontal plane, (b) the maximum stress in the beam.

4.199 Determine the maximum stress in each of the two machine elements shown.

4.200 The shape shown was formed by bending a thin steel plate. Assuming that the thickness $t$ is small compared to the length $a$ of a side of the shape, determine the stress (a) at A, (b) at B, (c) at C.
4.201 Three 120 × 10-mm steel plates have been welded together to form the beam shown. Assuming that the steel is elastoplastic with $E = 200$ GPa and $\sigma_Y = 300$ MPa, determine (a) the bending moment for which the plastic zones at the top and bottom of the beam are 40 mm thick, (b) the corresponding radius of curvature of the beam.

**Fig. P4.201**

4.202 A short column is made by nailing four 1 × 4-in. planks to a 4 × 4-in. timber. Determine the largest compressive stress created in the column by a 16-kip load applied as shown in the center of the top section of the timber if (a) the column is as described, (b) plank 1 is removed, (c) planks 1 and 2 are removed, (d) planks 1, 2, and 3 are removed, (e) all planks are removed.

**Fig. P4.202**

4.203 Two thin strips of the same material and same cross section are bent by couples of the same magnitude and glued together. After the two surfaces of contact have been securely bonded, the couples are removed. Denoting by $\sigma_1$ the maximum stress and by $\rho_1$ the radius of curvature of each strip while the couples were applied, determine (a) the final stresses at points $A$, $B$, $C$, and $D$, (b) the final radius of curvature.

**Fig. P4.203**
The following problems are designed to be solved with a computer.

4.C1 Two aluminum strips and a steel strip are to be bonded together to form a composite member of width \( b = 60 \) mm and depth \( h = 40 \) mm. The modulus of elasticity is 200 GPa for the steel and 75 GPa for the aluminum. Knowing that \( M = 1500 \) N \( \cdot \) m, write a computer program to calculate the maximum stress in the aluminum and in the steel for values of \( a \) from 0 to 20 mm using 2-mm increments. Using appropriate smaller increments, determine (a) the largest stress that can occur in the steel, (b) the corresponding value of \( a \).

4.C2 A beam of the cross section shown, made of a steel that is assumed to be elastoplastic with a yield strength \( \sigma_Y \) and a modulus of elasticity \( E \), is bent about the \( x \) axis. (a) Denoting by \( y_Y \) the half thickness of the elastic core, write a computer program to calculate the bending moment \( M \) and the radius of curvature \( r \) for values of \( y_Y \) from \( \frac{1}{2} d \) to \( \frac{3}{4} d \) using decrements equal to \( \frac{1}{2} t_f \). Neglect the effect of fillets. (b) Use this program to solve Prob. 4.201.

4.C3 An 8-kip \( \cdot \) in. couple \( M \) is applied to a beam of the cross section shown in a plane forming an angle \( \beta \) with the vertical. Noting that the centroid of the cross section is located at \( C \) and that the \( y \) and \( z \) axes are principal axes, write a computer program to calculate the stress at \( A \), \( B \), \( C \), and \( D \) for values of \( \beta \) from 0 to 180° using 10° increments. (Given: \( I_y = 6.23 \) in\(^4 \) and \( I_z = 1.481 \) in\(^4 \).)
**4.C4** Couples of moment \( M = 2 \text{ kN} \cdot \text{m} \) are applied as shown to a curved bar having a rectangular cross section with \( h = 100 \text{ mm} \) and \( b = 25 \text{ mm} \). Write a computer program and use it to calculate the stresses at points \( A \) and \( B \) for values of the ratio \( r/h \) from 10 to 1 using decrements of 1, and from 1 to 0.1 using decrements of 0.1. Using appropriate smaller increments, determine the ratio \( r/h \) for which the maximum stress in the curved bar is 50% larger than the maximum stress in a straight bar of the same cross section.

**4.C5** The couple \( M \) is applied to a beam of the cross section shown. 
(a) Write a computer program that, for loads expressed in either SI or U.S. customary units, can be used to calculate the maximum tensile and compressive stresses in the beam. 
(b) Use this program to solve Probs. 4.10, 4.11, and 4.19c.

**4.C6** A solid rod of radius \( c = 1.2 \text{ in.} \) is made of a steel that is assumed to be elastoplastic with \( E = 29,000 \text{ ksi} \) and \( \sigma_Y = 42 \text{ ksi} \). The rod is subjected to a couple of moment \( M \) that increases from zero to the maximum elastic moment \( M_E \) and then to the plastic moment \( M_p \). Denoting by \( y_Y \) the half thickness of the elastic core, write a computer program and use it to calculate the bending moment \( M \) and the radius of curvature \( p \) for values of \( y_Y \) from 1.2 in. to 0 using 0.2-in. decrements. (Hint: Divide the cross section into 80 horizontal elements of 0.03-in. height.)

**4.C7** The machine element of Prob. 4.18c is to be redesigned by removing part of the triangular cross section. It is believed that the removal of a small triangular area of width \( a \) will lower the maximum stress in the element. In order to verify this design concept, write a computer program to calculate the maximum stress in the element for values of \( a \) from 0 to 1 in. using 0.1-in. increments. Using appropriate smaller increments, determine the distance \( a \) for which the maximum stress is as small as possible and the corresponding value of the maximum stress.
The beams supporting the multiple overhead cranes system shown in this picture are subjected to transverse loads causing the beams to bend. The normal stresses resulting from such loadings will be determined in this chapter.
CHAPTER 5

Analysis and Design of Beams for Bending

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Chapter 5 Analysis and Design of Beams for Bending

5.1 Introduction
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5.1 INTRODUCTION

This chapter and most of the next one will be devoted to the analysis and the design of beams, i.e., structural members supporting loads applied at various points along the member. Beams are usually long, straight prismatic members, as shown in the photo on the previous page. Steel and aluminum beams play an important part in both structural and mechanical engineering. Timber beams are widely used in home construction (Photo 5.1). In most cases, the loads are perpendicular to the axis of the beam. Such a transverse loading causes only bending and shear in the beam. When the loads are not at a right angle to the beam, they also produce axial forces in the beam.

The transverse loading of a beam may consist of concentrated loads \( P_1, P_2, \ldots \), expressed in newtons, pounds, or their multiples, kilonewtons and kips (Fig. 5.1a), of a distributed load \( w \), expressed in N/m, kN/m, lb/ft, or kips/ft (Fig. 5.1b), or of a combination of both. When the load \( w \) per unit length has a constant value over part of the beam (as between \( A \) and \( B \) in Fig. 5.1b), the load is said to be uniformly distributed over that part of the beam.

Beams are classified according to the way in which they are supported. Several types of beams frequently used are shown in Fig. 5.2. The distance \( L \) shown in the various parts of the figure is...
called the span. Note that the reactions at the supports of the beams in parts \(a, b,\) and \(c\) of the figure involve a total of only three unknowns and, therefore, can be determined by the methods of statics. Such beams are said to be \textit{statically determinate} and will be discussed in this chapter and the next. On the other hand, the reactions at the supports of the beams in parts \(d, e,\) and \(f\) of Fig. 5.2 involve more than three unknowns and cannot be determined by the methods of statics alone. The properties of the beams with regard to their resistance to deformations must be taken into consideration. Such beams are said to be \textit{statically indeterminate} and their analysis will be postponed until Chap. 9, where deformations of beams will be discussed.

Sometimes two or more beams are connected by hinges to form a single continuous structure. Two examples of beams hinged at a point \(H\) are shown in Fig. 5.3. It will be noted that the reactions at the supports involve four unknowns and cannot be determined from the free-body diagram of the two-beam system. They can be determined, however, by recognizing that the internal moment at the hinge is zero. Then, after considering the free-body diagram of each beam separately, six unknowns are involved (including two force components at the hinge), and six equations are available.

When a beam is subjected to transverse loads, the internal forces in any section of the beam will generally consist of a shear force \(V\) and a bending couple \(M.\) Consider, for example, a simply supported beam \(AB\) carrying two concentrated loads and a uniformly distributed load (Fig. 5.4a). To determine the internal forces in a section through point \(C\) we first draw the free-body diagram of the entire beam to obtain the reactions at the supports (Fig. 5.4b). Passing a section through \(C\), we then draw the free-body diagram of \(AC\) (Fig. 5.4c), from which we determine the shear force \(V\) and the bending couple \(M.\)

The bending couple \(M\) creates \textit{normal stresses} in the cross section, while the shear force \(V\) creates \textit{shearing stresses} in that section. In most cases the dominant criterion in the design of a beam for strength is the maximum value of the normal stress in the beam. The determination of the normal stresses in a beam will be the subject of this chapter, while shearing stresses will be discussed in Chap. 6.

Since the distribution of the normal stresses in a given section depends only upon the value of the bending moment \(M\) in that section and the geometry of the section,† the elastic flexure formulas derived in Sec. 4.4 can be used to determine the maximum stress, as well as the stress at any given point, in the section. We write‡

\[
\sigma_n = \frac{Mc}{I} \quad \sigma_s = -\frac{My}{I} \quad (5.1, 5.2)
\]

†It is assumed that the distribution of the normal stresses in a given cross section is not affected by the deformations caused by the shearing stresses. This assumption will be verified in Sec. 6.5.

‡We recall from Sec. 4.2 that \(M\) can be positive or negative, depending upon whether the concavity of the beam at the point considered faces upward or downward. Thus, in the case considered here of a transverse loading, the sign of \(M\) can vary along the beam. On the other hand, since \(\sigma_n\) is a positive quantity, the absolute value of \(M\) is used in Eq. (5.1).
where \( I \) is the moment of inertia of the cross section with respect to a centroidal axis perpendicular to the plane of the couple, \( y \) is the distance from the neutral surface, and \( c \) is the maximum value of that distance (Fig. 4.11). We also recall from Sec. 4.4 that, introducing the elastic section modulus \( S = I/c \) of the beam, the maximum value \( \sigma_m \) of the normal stress in the section can be expressed as

\[
\sigma_m = \left| \frac{M}{S} \right|
\]

The fact that \( \sigma_m \) is inversely proportional to \( S \) underlines the importance of selecting beams with a large section modulus. Section moduli of various rolled-steel shapes are given in Appendix C, while the section modulus of a rectangular shape can be expressed, as shown in Sec. 4.4, as

\[
S = \frac{1}{6}bh^2
\]

where \( b \) and \( h \) are, respectively, the width and the depth of the cross section.

Equation (5.3) also shows that, for a beam of uniform cross section, \( \sigma_m \) is proportional to \( |M| \); Thus, the maximum value of the normal stress in the beam occurs in the section where \( |M| \) is largest. It follows that one of the most important parts of the design of a beam for a given loading condition is the determination of the location and magnitude of the largest bending moment.

The bending moment is determined if a bending-moment diagram is drawn, i.e., if the value of the bending moment \( M \) is determined at various points of the beam and plotted against the distance \( x \) measured from one end of the beam. It is further facilitated if a shear diagram is drawn at the same time by plotting the shear \( V \) against \( x \).

The sign convention to be used to record the values of the shear and bending moment will be discussed in Sec. 5.2. The values of \( V \) and \( M \) will then be obtained at various points of the beam by drawing free-body diagrams of successive portions of the beam. In Sec. 5.3 relations among load, shear, and bending moment will be derived and used to obtain the shear and bending-moment diagrams. This approach facilitates the determination of the largest absolute value of the bending moment and, thus, the determination of the maximum normal stress in the beam.

In Sec. 5.4 you will learn to design a beam for bending, i.e., so that the maximum normal stress in the beam will not exceed its allowable value. As indicated earlier, this is the dominant criterion in the design of a beam.

Another method for the determination of the maximum values of the shear and bending moment, based on expressing \( V \) and \( M \) in terms of singularity functions, will be discussed in Sec. 5.5. This approach lends itself well to the use of computers and will be expanded in Chap. 9 to facilitate the determination of the slope and deflection of beams.

Finally, the design of nonprismatic beams, i.e., beams with a variable cross section, will be discussed in Sec. 5.6. By selecting
the shape and size of the variable cross section so that its elastic section modulus $S = I/c$ varies along the length of the beam in the same way as $|M|$, it is possible to design beams for which the maximum normal stress in each section is equal to the allowable stress of the material. Such beams are said to be of constant strength.

5.2 SHEAR AND BENDING-MOMENT DIAGRAMS

As indicated in Sec. 5.1, the determination of the maximum absolute values of the shear and of the bending moment in a beam are greatly facilitated if $V$ and $M$ are plotted against the distance $x$ measured from one end of the beam. Besides, as you will see in Chap. 9, the knowledge of $M$ as a function of $x$ is essential to the determination of the deflection of a beam.

In the examples and sample problems of this section, the shear and bending-moment diagrams will be obtained by determining the values of $V$ and $M$ at selected points of the beam. These values will be found in the usual way, i.e., by passing a section through the point where they are to be determined (Fig. 5.5a) and considering the equilibrium of the portion of beam located on either side of the section (Fig. 5.5b). Since the shear forces $V$ and $V'$ have opposite senses, recording the shear at point $C$ with an up or down arrow would be meaningless, unless we indicated at the same time which of the free bodies $AC$ and $CB$ we are considering. For this reason, the shear will be recorded with a plus sign if the shearing forces are directed as shown in Fig. 5.5b, and a minus sign otherwise. A similar convention will apply for the bending moment $M$. It will be considered as positive if the bending couples are directed as shown in that figure, and negative otherwise.† Summarizing the sign conventions we have presented, we state:

The shear $V$ and the bending moment $M$ at a given point of a beam are said to be positive when the internal forces and couples acting on each portion of the beam are directed as shown in Fig. 5.6a.

These conventions can be more easily remembered if we note that

1. The shear at any given point of a beam is positive when the external forces (loads and reactions) acting on the beam tend to shear off the beam at that point as indicated in Fig. 5.6b.

†Note that this convention is the same that we used earlier in Sec. 4.2

Fig. 5.5 Determination of $V$ and $M$.

Fig. 5.6 Sign convention for shear and bending moment.
2. The bending moment at any given point of a beam is positive when the external forces acting on the beam tend to bend the beam at that point as indicated in Fig. 5.6c.

It is also of help to note that the situation described in Fig. 5.6, in which the values of the shear and of the bending moment are positive, is precisely the situation that occurs in the left half of a simply supported beam carrying a single concentrated load at its midpoint. This particular case is fully discussed in the next example.

**EXAMPLE 5.01**

Draw the shear and bending-moment diagrams for a simply supported beam AB of span L subjected to a single concentrated load P at its midpoint C (Fig. 5.7).

![Diagram of a simply supported beam with shear and bending-moment diagrams]

We first determine the reactions at the supports from the free-body diagram of the entire beam (Fig. 5.6a), we find that the magnitude of each reaction is equal to P/2.

Next we cut the beam at a point D between A and C and draw the free-body diagrams of AD and DB (Fig. 5.6b). Assuming that shear and bending moment are positive, we direct the internal forces V and V' and the internal couples M and M' as indicated in Fig. 5.6a. Considering the free body AD and writing that the sum of the vertical components and the sum of the moments about D of the forces acting on the free body are zero, we find V = +P/2 and M = +P's/2. Both the shear and the bending moment are therefore positive; this may be checked by observing that the reaction at A tends to shear off and to bend the beam at D as indicated in Figs. 5.6b and c. We now plot V and M between A and C (Figs. 5.6d and e); the shear has a constant value V = P/2, while the bending moment increases linearly from M = 0 at x = 0 to M = PL/4 at x = L/2.

Cutting, now, the beam at a point E between C and B and considering the free body EB (Fig. 5.6c), we write that the sum of the vertical components and the sum of the moments about E of the forces acting on the free body are zero. We obtain V = −P/2 and M = P(L − x)/2. The shear is therefore negative and the bending moment positive; this can be checked by observing that the reaction at B bends the beam at E as indicated in Fig. 5.6c but tends to shear it off in a manner opposite to that shown in Fig. 5.6b. We can complete, now, the shear and bending-moment diagrams of Figs. 5.6d and e; the shear has a constant value V = −P/2 between C and B, while the bending moment decreases linearly from M = PL/4 at x = L/2 to M = 0 at x = L.
We note from the foregoing example that, when a beam is subjected only to concentrated loads, the shear is constant between loads and the bending moment varies linearly between loads. In such situations, therefore, the shear and bending-moment diagrams can easily be drawn, once the values of \( V \) and \( M \) have been obtained at sections selected just to the left and just to the right of the points where the loads and reactions are applied (see Sample Prob. 5.1).

**EXAMPLE 5.02**

Draw the shear and bending-moment diagrams for a cantilever beam \( AB \) of span \( L \) supporting a uniformly distributed load \( w \) (Fig. 5.9).

![Fig. 5.9](image_url)

We cut the beam at a point \( C \) between \( A \) and \( B \) and draw the free-body diagram of \( AC \) (Fig. 5.10a), directing \( V \) and \( M \) as indicated in Fig. 5.6a. Denoting by \( x \) the distance from \( A \) to \( C \) and replacing the distributed load over \( AC \) by its resultant \( wx \) applied at the midpoint of \( AC \), we write

\[

\begin{align*}
+ \sum F_y &= 0: \quad -wx - V = 0 \quad V = -wx \\
+ \sum M_c &= 0: \quad wx \left( \frac{x}{2} \right) + M = 0 \quad M = -\frac{1}{2}wx^2
\end{align*}

\]

We note that the shear diagram is represented by an oblique straight line (Fig. 5.10b) and the bending-moment diagram by a parabola (Fig. 5.10c). The maximum values of \( V \) and \( M \) both occur at \( B \), where we have

\[
\begin{align*}
V_B &= -wL \quad M_B = -\frac{1}{2}wL^2
\end{align*}
\]

![Fig. 5.10](image_url)
SAMPLE PROBLEM 5.1

For the timber beam and loading shown, draw the shear and bending-moment diagrams and determine the maximum normal stress due to bending.

SOLUTION

Reactions. Considering the entire beam as a free body, we find

\[ R_B = 40 \text{ kN} \uparrow \quad R_D = 14 \text{ kN} \uparrow \]

Shear and Bending-Moment Diagrams. We first determine the internal forces just to the right of the 20-kN load at A. Considering the stub of beam to the left of section 1 as a free body and assuming V and M to be positive (according to the standard convention), we write

\[ +\Sigma F_y = 0: \quad -20 \text{ kN} - V_1 = 0 \quad V_1 = -20 \text{ kN} \]
\[ +\Sigma M_1 = 0: \quad (20 \text{ kN})(0 \text{ m}) + M_1 = 0 \quad M_1 = 0 \]

We next consider as a free body the portion of beam to the left of section 2 and write

\[ +\Sigma F_y = 0: \quad -20 \text{ kN} - V_2 = 0 \quad V_2 = -20 \text{ kN} \]
\[ +\Sigma M_2 = 0: \quad (20 \text{ kN})(2.5 \text{ m}) + M_2 = 0 \quad M_2 = -50 \text{ kN} \cdot \text{m} \]

The shear and bending moment at sections 3, 4, 5, and 6 are determined in a similar way from the free-body diagrams shown. We obtain

\[ V_3 = +26 \text{ kN} \quad M_3 = -50 \text{ kN} \cdot \text{m} \]
\[ V_4 = +26 \text{ kN} \quad M_4 = +28 \text{ kN} \cdot \text{m} \]
\[ V_5 = -14 \text{ kN} \quad M_5 = +28 \text{ kN} \cdot \text{m} \]
\[ V_6 = -14 \text{ kN} \quad M_6 = 0 \]

For several of the latter sections, the results may be more easily obtained by considering as a free body the portion of the beam to the right of the section. For example, for the portion of the beam to the right of section 4, we have

\[ +\Sigma F_y = 0: \quad V_4 - 40 \text{ kN} + 14 \text{ kN} = 0 \quad V_4 = +26 \text{ kN} \]
\[ +\Sigma M_4 = 0: \quad -M_4 + (14 \text{ kN})(2 \text{ m}) = 0 \quad M_4 = +28 \text{ kN} \cdot \text{m} \]

We can now plot the six points shown on the shear and bending-moment diagrams. As indicated earlier in this section, the shear is of constant value between concentrated loads, and the bending moment varies linearly; we obtain therefore the shear and bending-moment diagrams shown.

Maximum Normal Stress. It occurs at B, where |M| is largest. We use Eq. (5.4) to determine the section modulus of the beam:

\[ S = \frac{bh^2}{6} = \frac{(0.080 \text{ m})(0.250 \text{ m})^2}{6} = 833.33 \times 10^{-6} \text{ m}^3 \]

Substituting this value and |M| = |M_B| = 50 \times 10^3 \text{ N} \cdot \text{m} into Eq. (5.3) gives

\[ \sigma_m = \frac{|M_B|}{S} = \frac{(50 \times 10^3 \text{ N} \cdot \text{m})}{833.33 \times 10^{-6}} = 60.00 \times 10^6 \text{ Pa} \]

Maximum normal stress in the beam = 60.0 MPa
SAMPLE PROBLEM 5.2

The structure shown consists of a W10 × 112 rolled-steel beam AB and of two short members welded together and to the beam. (a) Draw the shear and bending-moment diagrams for the beam and the given loading. (b) Determine the maximum normal stress in sections just to the left and just to the right of point D.

SOLUTION

Equivalent Loading of Beam. The 10-kip load is replaced by an equivalent force-couple system at D. The reaction at B is determined by considering the beam as a free body.

a. Shear and Bending-Moment Diagrams

From A to C. We determine the internal forces at a distance x from point A by considering the portion of beam to the left of section J. That part of the distributed load acting on the free body is replaced by its resultant, and we write

\[ + \sum F_y = 0: \quad -3x - V = 0 \quad V = -3x \text{ kips} \]
\[ + \sum M_1 = 0: \quad 3x(\frac{1}{2}x) + M = 0 \quad M = -1.5x^2 \text{ kip } \cdot \text{ ft} \]

Since the free-body diagram shown can be used for all values of x smaller than 8 ft, the expressions obtained for V and M are valid in the region 0 ≤ x < 8 ft.

From C to D. Considering the portion of beam to the left of section 2 and again replacing the distributed load by its resultant, we obtain

\[ + \sum F_y = 0: \quad -24 - V = 0 \quad V = -24 \text{ kips} \]
\[ + \sum M_2 = 0: \quad 24(x - 4) + M = 0 \quad M = 96 - 24x \text{ kip } \cdot \text{ ft} \]

These expressions are valid in the region 8 ft < x < 11 ft.

From D to B. Using the position of beam to the left of section 3, we obtain for the region 11 ft < x < 16 ft

\[ V = -34 \text{ kips} \quad M = 226 - 34x \text{ kip } \cdot \text{ ft} \]

The shear and bending-moment diagrams for the entire beam can now be plotted. We note that the couple of moment 20 kip · ft applied at point D introduces a discontinuity into the bending-moment diagram.

b. Maximum Normal Stress to the Left and Right of Point D. From Appendix C we find that for the W10 × 112 rolled-steel shape, \( S = 126 \text{ in}^3 \) about the X-X axis.

To the left of D: We have \( |M| = 168 \text{ kip } \cdot \text{ ft} = 2016 \text{ kip } \cdot \text{ in} \). Substituting for \( |M| \) and \( S \) into Eq. (5.3), we write

\[ \sigma_{\text{m left}} = \frac{|M|}{S} = \frac{2016 \text{ kip } \cdot \text{ in}}{126 \text{ in}^3} = 16.00 \text{ ksi} \quad \sigma_{\text{m left}} = 16.00 \text{ ksi} \]

To the right of D: We have \( |M| = 148 \text{ kip } \cdot \text{ ft} = 1776 \text{ kip } \cdot \text{ in} \). Substituting for \( |M| \) and \( S \) into Eq. (5.3), we write

\[ \sigma_{\text{m right}} = \frac{|M|}{S} = \frac{1776 \text{ kip } \cdot \text{ in}}{126 \text{ in}^3} = 14.10 \text{ ksi} \quad \sigma_{\text{m right}} = 14.10 \text{ ksi} \]
**PROBLEMS**

**5.1 through 5.6** For the beam and loading shown, *(a)* draw the shear and bending-moment diagrams, *(b)* determine the equations of the shear and bending-moment curves.

![Fig. P5.1](image1) \[ A \quad B \quad C \]

![Fig. P5.2](image2) \[ A \quad B \quad C \]

![Fig. P5.3](image3) \[ A \quad B \quad C \]

![Fig. P5.4](image4) \[ A \quad B \quad C \]

![Fig. P5.5](image5) \[ A \quad B \quad C \]

![Fig. P5.6](image6) \[ A \quad B \quad C \]

**5.7 and 5.8** Draw the shear and bending-moment diagrams for the beam and loading shown, and determine the maximum absolute value *(a)* of the shear, *(b)* of the bending moment.

![Fig. P5.7](image7) \[ A \quad 300 \quad 240 \quad 360 \quad B \]

![Fig. P5.8](image8) \[ A \quad 200 \quad 200 \quad 500 \quad 200 \quad B \]

Dimensions in mm
**5.9 and 5.10** Draw the shear and bending-moment diagrams for the beam and loading shown, and determine the maximum absolute value (a) of the shear, (b) of the bending moment.

![Fig. P5.9](image)

![Fig. P5.10](image)

**5.11 and 5.12** Draw the shear and bending-moment diagrams for the beam and loading shown, and determine the maximum absolute value (a) of the shear, (b) of the bending moment.

![Fig. P5.11](image)

![Fig. P5.12](image)

**5.13 and 5.14** Assuming that the tension of the ground is uniformly distributed, draw the shear and bending-moment diagrams for the beam AB and determine the maximum absolute value (a) of the shear, (b) of the bending moment.

![Fig. P5.13](image)

![Fig. P5.14](image)

**5.15 and 5.16** For the beam and loading shown, determine the maximum normal stress due to bending on a transverse section at C.

![Fig. P5.15](image)

![Fig. P5.16](image)
5.17 For the beam and loading shown, determine the maximum normal stress due to bending on a transverse section at $C$.

![Diagram of beam with loads and dimensions](image)

**Fig. P5.17**

5.18 For the beam and loading shown, determine the maximum normal stress due to bending on section $a-a$.

![Diagram of beam with loads and dimensions](image)

**Fig. P5.18**

5.19 For the beam and loading shown, determine the maximum normal stress due to bending on a transverse section at $C$.

![Diagram of beam with loads and dimensions](image)

**Fig. P5.19**

5.20 For the beam and loading shown and determine the maximum normal stress due to bending.

![Diagram of beam with loads and dimensions](image)

**Fig. P5.20**

5.21 Draw the shear and bending-moment diagrams for the beam and loading shown and determine the maximum normal stress due to bending.

![Diagram of beam with loads and dimensions](image)

**Fig. P5.21**
5.22 and 5.23 Draw the shear and bending-moment diagrams for the beam and loading shown and determine the maximum normal stress due to bending.

5.24 and 5.25 Draw the shear and bending-moment diagrams for the beam and loading shown and determine the maximum normal stress due to bending.

5.26 Knowing that $W = 12$ kN, draw the shear and bending-moment diagrams for beam $AB$ and determine the maximum normal stress due to bending.

5.27 Determine (a) the magnitude of the counterweight $W$ for which the maximum absolute value of the bending moment in the beam is as small as possible, (b) the corresponding maximum normal stress due to bending. (Hint: Draw the bending-moment diagram and equate the absolute values of the largest positive and negative bending moments obtained.)

5.28 Determine (a) the distance $a$ for which the absolute value of the bending moment in the beam is as small as possible, (b) the corresponding maximum normal stress due to bending. (See hint of Prob. 5.27.)
5.29 Determine (a) the distance \( a \) for which the absolute value of the bending moment in the beam is as small as possible, (b) the corresponding maximum normal stress due to bending. (See hint of Prob. 5.27.)

\[
\begin{array}{c}
A \\
\downarrow 0.8 \text{kips} \\
C \\
\downarrow 1.2 \text{kips} \\
D \\
\downarrow 1.2 \text{kips} \\
E \\
\downarrow \\
B \\
\end{array}
\]

\( a \) = 1.5 ft, 1.2 ft, 0.9 ft

Fig. P5.29

5.30 Knowing that \( P = Q = 480 \text{ N} \), determine (a) the distance \( a \) for which the absolute value of the bending moment in the beam is as small as possible, (b) the corresponding maximum normal stress due to bending. (See hint of Prob. 5.27.)

\[
\begin{array}{c}
A \\
\downarrow 500 \text{ mm} \\
C \\
\downarrow 500 \text{ mm} \\
D \\
\downarrow 12 \text{ mm} \\
B \\
\end{array}
\]

\( a \) = 12 mm, 18 mm

Fig. P5.30

5.31 Solve Prob. 5.30, assuming that \( P = 480 \text{ N} \) and \( Q = 320 \text{ N} \).

5.32 A solid steel bar has a square cross section of side \( b \) and is supported as shown. Knowing that for steel \( \rho = 7860 \text{ kg/m}^3 \), determine the dimension \( b \) for which the maximum normal stress due to bending is (a) 10 MPa, (b) 50 MPa.

\[
\begin{array}{c}
A \\
\downarrow 1.2 \text{ m} \\
C \\
\downarrow 1.2 \text{ m} \\
D \\
\downarrow 1.2 \text{ m} \\
B \\
\end{array}
\]

\( b \)

Fig. P5.32

5.33 A solid steel rod of diameter \( d \) is supported as shown. Knowing that for steel \( \gamma = 490 \text{ lb/ft}^3 \), determine the smallest diameter \( d \) that can be used if the normal stress due to bending is not to exceed 4 ksi.

\[
\begin{array}{c}
A \\
\downarrow L = 10 \text{ ft} \\
B \\
\end{array}
\]

\( d \)

Fig. P5.33
5.3 RELATIONS AMONG LOAD, SHEAR, AND BENDING MOMENT

When a beam carries more than two or three concentrated loads, or when it carries distributed loads, the method outlined in Sec. 5.2 for plotting shear and bending moment can prove quite cumbersome. The construction of the shear diagram and, especially, of the bending-moment diagram will be greatly facilitated if certain relations existing among load, shear, and bending moment are taken into consideration.

Let us consider a simply supported beam AB carrying a distributed load \( w \) per unit length (Fig. 5.11a), and let C and C′ be two points of the beam at a distance \( \Delta x \) from each other. The shear and bending moment at C will be denoted by \( V \) and \( M \), respectively, and will be assumed positive; the shear and bending moment at C′ will be denoted by \( V' \) and \( M' \). We now detach the portion of beam CC′ and draw its free-body diagram (Fig. 5.11b). The forces exerted on the free body include a load of magnitude \( w \Delta x \) and internal forces and couples at C and C′. Since shear and bending moment have been assumed positive, the forces and couples will be directed as shown in the figure.

**Relations between Load and Shear.** Writing that the sum of the vertical components of the forces acting on the free body CC′ is zero, we have

\[
\Delta V - (V + \Delta V) = -w \Delta x
\]

Dividing both members of the equation by \( \Delta x \) and then letting \( \Delta x \) approach zero, we obtain

\[
\frac{dV}{dx} = -w
\]  

Equation (5.5) indicates that, for a beam loaded as shown in Fig. 5.11a, the slope \( dV/dx \) of the shear curve is negative; the numerical value

\[
\frac{w \Delta x}{\frac{1}{2} \Delta x}
\]

**Fig. 5.11** Simply supported beam subjected to a distributed load.
of the slope at any point is equal to the load per unit length at that point.

Integrating (5.5) between points C and D, we write

\[ V_D - V_C = - \int_{x_c}^{x_0} w \, dx \] (5.6)

\[ V_D - V_C = -(\text{area under load curve between } C \text{ and } D) \] (5.6')

Note that this result could also have been obtained by considering the equilibrium of the portion of beam CD, since the area under the load curve represents the total load applied between C and D.

It should be observed that Eq. (5.5) is not valid at a point where a concentrated load is applied; the shear curve is discontinuous at such a point, as seen in Sec. 5.2. Similarly, Eqs. (5.6) and (5.6') cease to be valid when concentrated loads are applied between C and D, since they do not take into account the sudden change in shear caused by a concentrated load. Equations (5.6) and (5.6'), therefore, should be applied only between successive concentrated loads.

**Relations between Shear and Bending Moment.** Returning to the free-body diagram of Fig. 5.11b, and writing now that the sum of the moments about C' is zero, we have

\[ + \Sigma M_M = 0: \quad (M + \Delta M) - M - V \Delta x + w \Delta x \frac{\Delta x}{2} = 0 \]

\[ \Delta M = V \Delta x - \frac{1}{2} w (\Delta x)^2 \]

Dividing both members of the equation by \( \Delta x \) and then letting \( \Delta x \) approach zero, we obtain

\[ \frac{dM}{dx} = V \] (5.7)

Equation (5.7) indicates that the slope \( dM/dx \) of the bending-moment curve is equal to the value of the shear. This is true at any point where the shear has a well-defined value, i.e., at any point where no concentrated load is applied. Equation (5.7) also shows that \( V = 0 \) at points where \( M \) is maximum. This property facilitates the determination of the points where the beam is likely to fail under bending.

Integrating (5.7) between points C and D, we write

\[ M_D - M_C = \int_{x_c}^{x_0} V \, dx \] (5.8)

\[ M_D - M_C = \text{area under shear curve between } C \text{ and } D \] (5.8')
Note that the area under the shear curve should be considered positive where the shear is positive and negative where the shear is negative. Equations (5.8) and (5.8') are valid even when concentrated loads are applied between C and D, as long as the shear curve has been correctly drawn. The equations cease to be valid, however, if a couple is applied at a point between C and D, since they do not take into account the sudden change in bending moment caused by a couple (see Sample Prob. 5.6).

**EXAMPLE 5.03**

Draw the shear and bending-moment diagrams for the simply supported beam shown in Fig. 5.12 and determine the maximum value of the bending moment.

From the free-body diagram of the entire beam, we determine the magnitude of the reactions at the supports.

\[ R_A = R_B = \frac{1}{2} wL \]

Next, we draw the shear diagram. Close to the end A of the beam, the shear is equal to \( R_A \), that is, to \( \frac{1}{2} wL \), as we can check by considering as a free body a very small portion of the beam. Using Eq. (5.6), we determine the shear \( V \) at any distance \( x \) from A; we write

\[ V = V_A - wx = \frac{1}{2} wL - wx = w\left(\frac{1}{2}L - x\right) \]

The shear curve is thus an oblique straight line which crosses the x axis at \( x = L/2 \) (Fig. 5.13(a)). Considering, now, the bending moment, we first observe that \( M_A = 0 \). The value \( M \) of the bending moment at any distance \( x \) from A may then be obtained from Eq. (5.8); we have

\[ M - M_A = \int_0^x V \, dx \]

\[ M = \int_0^x w\left(\frac{1}{2}L - x\right) dx = \frac{1}{2} wL (Lx - x^2) \]

The bending-moment curve is a parabola. The maximum value of the bending moment occurs when \( x = L/2 \), since \( V \) (and thus \( dM/dx \)) is zero for that value of \( x \). Substituting \( x = L/2 \) in the last equation, we obtain \( M_{\text{max}} = wL^2/8 \) (Fig. 5.13(b)).
In most engineering applications, one needs to know the value of the bending moment only at a few specific points. Once the shear diagram has been drawn, and after \( M \) has been determined at one of the ends of the beam, the value of the bending moment can then be obtained at any given point by computing the area under the shear curve and using Eq. (5.8). For instance, since \( M_A = 0 \) for the beam of Example 5.03, the maximum value of the bending moment for that beam can be obtained simply by measuring the area of the shaded triangle in the shear diagram of Fig. 5.13a.

We have

\[
M_{\text{max}} = \frac{1}{2} L \frac{wL}{2} = \frac{wL^2}{8}
\]

We note that, in this example, the load curve is a horizontal straight line, the shear curve an oblique straight line, and the bending-moment curve a parabola. If the load curve had been an oblique straight line (first degree), the shear curve would have been a parabola (second degree) and the bending-moment curve a cubic (third degree). The shear and bending-moment curves will always be, respectively, one and two degrees higher than the load curve. With this in mind, we should be able to sketch the shear and bending-moment diagrams without actually determining the functions \( V(x) \) and \( M(x) \), once a few values of the shear and bending moment have been computed. The sketches obtained will be more accurate if we make use of the fact that, at any point where the curves are continuous, the slope of the shear curve is equal to \(-w\) and the slope of the bending-moment curve is equal to \(V\).
SAMPLE PROBLEM 5.3

Draw the shear and bending-moment diagrams for the beam and loading shown.

SOLUTION

Reactions. Considering the entire beam as a free body, we write

\[ + \Sigma M_A = 0: \]
\[ D(24 \text{ ft}) - (20 \text{ kips})(6 \text{ ft}) - (12 \text{ kips})(14 \text{ ft}) - (12 \text{ kips})(28 \text{ ft}) = 0 \]
\[ D = +26 \text{ kips} \]
\[ \Sigma F_y = 0: \]
\[ A_y - 20 \text{ kips} - 12 \text{ kips} + 26 \text{ kips} - 12 \text{ kips} = 0 \]
\[ A_y = +18 \text{ kips} \]
\[ \Sigma F_x = 0: \]
\[ A_x = 0 \]

We also note that at both A and E the bending moment is zero; thus, two points (indicated by dots) are obtained on the bending-moment diagram.

Shear Diagram. Since \( dV/dx = -w \), we find that between concentrated loads and reactions the slope of the shear diagram is zero (i.e., the shear is constant). The shear at any point is determined by dividing the beam into two parts and considering either part as a free body. For example, using the portion of beam to the left of section I, we obtain the shear between B and C:

\[ 18 \text{ kips} - 20 \text{ kips} - V = 0 \]
\[ V = -2 \text{ kips} \]

We also find that the shear is +12 kips just to the right of D and zero at end E. Since the slope \( dV/dx = -w \) is constant between D and E, the shear diagram between these two points is a straight line.

Bending-Moment Diagram. We recall that the area under the shear curve between two points is equal to the change in bending moment between the same two points. For convenience, the area of each portion of the shear diagram is computed and is indicated in parentheses on the diagram. Since the bending moment \( M_A \) at the left end is known to be zero, we write

\[ M_B - M_A = +108 \text{ kip} \cdot \text{ft} \]
\[ M_C - M_B = -16 \text{ kip} \cdot \text{ft} \]
\[ M_D - M_C = -140 \text{ kip} \cdot \text{ft} \]
\[ M_E - M_D = +48 \text{ kip} \cdot \text{ft} \]

Since \( M_E \) is known to be zero, a check of the computations is obtained.

Between the concentrated loads and reactions, the shear is constant; thus, the slope \( dM/dx \) is constant, and the bending-moment diagram is drawn by connecting the known points with straight lines. Between D and E where the shear diagram is an oblique straight line, the bending-moment diagram is a parabola.

From the V and M diagrams we note that \( V_{\text{max}} = 18 \text{ kips} \) and \( M_{\text{max}} = 108 \text{ kip} \cdot \text{ft} \).
SAMPLE PROBLEM 5.4

The W360 × 79 rolled-steel beam AC is simply supported and carries the uniformly distributed load shown. Draw the shear and bending-moment diagrams for the beam and determine the location and magnitude of the maximum normal stress due to bending.

SOLUTION

Reactions. Considering the entire beam as a free body, we find

\[ R_A = 80 \text{ kN} \uparrow \quad R_C = 40 \text{ kN} \uparrow \]

Shear Diagram. The shear just to the right of A is \( V_A = +80 \text{ kN} \).

Since the change in shear between two points is equal to minus the area under the load curve between the same two points, we obtain \( V_B \) by writing

\[ V_B - V_A = -(20 \text{ kN/m})(6 \text{ m}) = -120 \text{ kN} \]

\[ V_B = -120 + V_A = -120 + 80 = -40 \text{ kN} \]

The slope \( dV/dx = -w \) being constant between A and B, the shear diagram between these two points is represented by a straight line. Between B and C, the area under the load curve is zero; therefore,

\[ V_C - V_B = 0 \quad V_C = V_B = -40 \text{ kN} \]

and the shear is constant between B and C.

Bending-Moment Diagram. We note that the bending moment at each end of the beam is zero. In order to determine the maximum bending moment, we locate the section D of the beam where \( V = 0 \). We write

\[ V_D - V_A = -wx \]

\[ 0 - 80 \text{ kN} = -(20 \text{ kN/m})x \]

and, solving for \( x \) we find:

\[ x = 4 \text{ m} \]

The maximum bending moment occurs at point D, where we have \( dM/dx = V = 0 \). The areas of the various portions of the shear diagram are computed and are given (in parentheses) on the diagram. Since the area of the shear diagram between two points is equal to the change in bending moment between the same two points, we write

\[ M_D - M_A = +160 \text{ kN} \cdot \text{m} \quad M_D = +160 \text{ kN} \cdot \text{m} \]

\[ M_B - M_D = -40 \text{ kN} \cdot \text{m} \quad M_B = +120 \text{ kN} \cdot \text{m} \]

\[ M_C - M_B = -120 \text{ kN} \cdot \text{m} \quad M_C = 0 \]

The bending-moment diagram consists of an arc of parabola followed by a segment of straight line; the slope of the parabola at A is equal to the value of \( V \) at that point.

Maximum Normal Stress. It occurs at D, where \( |M| \) is largest. From Appendix C we find that for a W360 × 79 rolled-steel shape, \( S = 1270 \text{ mm}^3 \) about a horizontal axis. Substituting this value and \( |M| = |M_D| = 160 \times 10^3 \text{ N} \cdot \text{m} \) into Eq. 5.3, we write

\[ \sigma_m = \frac{|M_D|}{S} = \frac{160 \times 10^3 \text{ N} \cdot \text{m}}{1270 \times 10^{-6} \text{ m}^3} = 126.0 \times 10^6 \text{ Pa} \]

Maximum normal stress in the beam = 126.0 MPa
SAMPLE PROBLEM 5.5

Sketch the shear and bending-moment diagrams for the cantilever beam shown.

SOLUTION

Shear Diagram. At the free end of the beam, we find $V_A = 0$. Between $A$ and $B$, the area under the load curve is $\frac{1}{2}w_0a$; we find $V_B$ by writing $V_B = 2V_A = \frac{1}{2}w_0a$. Between $B$ and $C$, the beam is not loaded; thus $V_C = V_B$. At $A$, we have $w = w_0$ and, according to Eq. (5.5), the slope of the shear curve is $dV_y/dx = -w_0$, while at $B$ the slope is $dV_y/dx = 0$. Between $A$ and $B$, the loading decreases linearly, and the shear diagram is parabolic. Between $B$ and $C$, $w = 0$, and the shear diagram is a horizontal line.

Bending-Moment Diagram. The bending moment $M_A$ at the free end of the beam is zero. We compute the area under the shear curve and write $M_B = M_A = -\frac{1}{2}w_0a^2$. Between $B$ and $C$, the beam is not loaded; thus $M_C = M_B$. At $A$, we have $w = w_0$ and, according to Eq. (5.5), the slope of the shear curve is $dV_y/dx = -w_0$, while at $B$ the slope is $dV_y/dx = 0$. Between $A$ and $B$, the loading decreases linearly, and the shear diagram is parabolic. Between $B$ and $C$, $w = 0$, and the shear diagram is a horizontal line.

SAMPLE PROBLEM 5.6

The simple beam $AC$ is loaded by a couple of moment $T$ applied at point $B$. Draw the shear and bending-moment diagrams of the beam.

SOLUTION

The entire beam is taken as a free body, and we obtain $R_A = \frac{T}{L} \uparrow \quad R_C = \frac{T}{L} \downarrow$

The shear at any section is constant and equal to $T/L$. Since a couple is applied at $B$, the bending-moment diagram is discontinuous at $B$; it is represented by two oblique straight lines and decreases suddenly at $B$ by an amount equal to $T$. The character of this discontinuity can also be verified by equilibrium analysis. For example, considering the free body of the portion of the beam from $A$ to just beyond the right of $B$ as shown, we find the value of $M$ by $+\sum M_B = 0: \quad -\frac{T}{L}a + T + M = 0 \quad M = -T\left(1 - \frac{a}{L}\right)$
5.34 Using the method of Sec. 5.3, solve Prob. 5.1a.

5.35 Using the method of Sec. 5.3, solve Prob. 5.2a.

5.36 Using the method of Sec. 5.3, solve Prob. 5.3a.

5.37 Using the method of Sec. 5.3, solve Prob. 5.4a.

5.38 Using the method of Sec. 5.3, solve Prob. 5.5a.

5.39 Using the method of Sec. 5.3, solve Prob. 5.6a.

5.40 Using the method of Sec. 5.3, solve Prob. 5.7.

5.41 Using the method of Sec. 5.3, solve Prob. 5.8.

5.42 Using the method of Sec. 5.3, solve Prob. 5.9.

5.43 Using the method of Sec. 5.3, solve Prob. 5.10.

5.44 and 5.45 Draw the shear and bending-moment diagrams for the beam and loading shown, and determine the maximum absolute value (a) of the shear, (b) of the bending moment.

5.46 Using the method of Sec. 5.3, solve Prob. 5.15.

5.47 Using the method of Sec. 5.3, solve Prob. 5.16.

5.48 Using the method of Sec. 5.3, solve Prob. 5.18.

5.49 Using the method of Sec. 5.3, solve Prob. 5.19.

5.50 For the beam and loading shown, determine the equations of the shear and bending-moment curves and the maximum absolute value of the bending moment in the beam, knowing that (a) \( k = 1 \), (b) \( k = 0.5 \).
5.51 and 5.52 Determine (a) the equations of the shear and bending-moment curves for the beam and loading shown, (b) the maximum absolute value of the bending moment in the beam.

\[ w = w_0 \frac{x}{L} \quad \text{Fig. P5.51} \]

\[ w = w_0 \sin \frac{\pi x}{L} \quad \text{Fig. P5.52} \]

5.53 Determine (a) the equations of the shear and bending-moment curves for the beam and loading shown, (b) the maximum absolute value of the bending moment in the beam.

\[ w = w_0 \cos \frac{\pi x}{2L} \quad \text{Fig. P5.53} \]

5.54 and 5.55 Draw the shear and bending-moment diagrams for the beam and loading shown and determine the maximum normal stress due to bending.

\[ \text{Fig. P5.54} \]

5.56 and 5.57 Draw the shear and bending-moment diagrams for the beam and loading shown and determine the maximum normal stress due to bending.

\[ \text{Fig. P5.56} \]

\[ \text{Fig. P5.57} \]
5.58 and 5.59 Draw the shear and bending-moment diagrams for the beam and loading shown and determine the maximum normal stress due to bending.

Fig. P5.58

Fig. P5.59

5.60 Beam AB, of length L and square cross section of side a, is supported by a pivot at C and loaded as shown. (a) Check that the beam is in equilibrium. (b) Show that the maximum stress due to bending occurs at C and is equal to \( w_0 L^2 / (1.5a)^3 \).

Fig. P5.60

5.61 Knowing that beam AB is in equilibrium under the loading shown, draw the shear and bending-moment diagrams and determine the maximum normal stress due to bending.

Fig. P5.61

5.62 The beam AB supports a uniformly distributed load of 480 lb/ft and two concentrated loads P and Q. The normal stress due to bending on the bottom edge of the lower flange is +14.85 ksi at D and +10.65 ksi at E. (a) Draw the shear and bending-moment diagrams for the beam. (b) Determine the maximum normal stress due to bending that occurs in the beam.

Fig. P5.62
**5.63** Beam AB supports a uniformly distributed load of 2 kN/m and two concentrated loads P and Q. It has been experimentally determined that the normal stress due to bending in the bottom edge of the beam is −56.9 MPa at A and −29.9 MPa at C. Draw the shear and bending-moment diagrams for the beam and determine the magnitudes of the loads P and Q.

**5.64** The beam AB supports two concentrated loads P and Q. The normal stress due to bending on the bottom edge of the beam is +55 MPa at D and +37.5 MPa at F. (a) Draw the shear and bending-moment diagrams for the beam. (b) Determine the maximum normal stress due to bending that occurs in the beam.

---

5.4 DESIGN OF PRISMATIC BEAMS FOR BENDING

As indicated in Sec. 5.1, the design of a beam is usually controlled by the maximum absolute value $|M|_{\text{max}}$ of the bending moment that will occur in the beam. The largest normal stress $\sigma_m$ in the beam is found at the surface of the beam in the critical section where $|M|_{\text{max}}$ occurs and can be obtained by substituting $|M|_{\text{max}}$ for $|M|$ in Eq. (5.1) or Eq. (5.3).† We write

$$\sigma_m = \frac{|M|_{\text{max}} c}{I} \quad \sigma_m = \frac{|M|_{\text{max}}}{S} \quad (5.1', \ 5.3')$$

A safe design requires that $\sigma_m \leq \sigma_{\text{all}}$, where $\sigma_{\text{all}}$ is the allowable stress for the material used. Substituting $\sigma_{\text{all}}$ for $\sigma_m$ in (5.3') and solving for $S$ yields the minimum allowable value of the section modulus for the beam being designed:

$$S_{\text{min}} = \frac{|M|_{\text{max}}}{\sigma_{\text{all}}} \quad (5.9)$$

The design of common types of beams, such as timber beams of rectangular cross section and rolled-steel beams of various cross-sectional shapes, will be considered in this section. A proper procedure should lead to the most economical design. This means that, among beams of the same type and the same material, and other

†For beams that are not symmetrical with respect to their neutral surface, the largest of the distances from the neutral surface to the surfaces of the beam should be used for $c$ in Eq. (5.1) and in the computation of the section modulus $S = I/c$. 
things being equal, the beam with the smallest weight per unit
length—and, thus, the smallest cross-sectional area—should be
selected, since this beam will be the least expensive.

The design procedure will include the following steps†:

1. First determine the value of $\sigma_{\text{all}}$ for the material selected from
   a table of properties of materials or from design specifications.
   You can also compute this value by dividing the ultimate strength
   $\sigma_U$ of the material by an appropriate factor of safety (Sec. 1.13).
   Assuming for the time being that the value of $\sigma_{\text{all}}$ is the same
   in tension and in compression, proceed as follows.

2. Draw the shear and bending-moment diagrams corresponding
   to the specified loading conditions, and determine the maximum
   absolute value $|M|_{\text{max}}$ of the bending moment in the beam.

3. Determine from Eq. (5.9) the minimum allowable value $S_{\text{min}}$
   of the section modulus of the beam.

4. For a timber beam, the depth $h$ of the beam, its width $b$, or
   the ratio $h/b$ characterizing the shape of its cross section will
   probably have been specified. The unknown dimensions may
   then be selected by recalling from Eq. (4.19) of Sec. 4.4 that
   $b$ and $h$ must satisfy the relation $\frac{1}{6}bh^2 = S \geq S_{\text{min}}$.

5. For a rolled-steel beam, consult the appropriate table in Appen-
   dix C. Of the available beam sections, consider only those with a
   section modulus $S \geq S_{\text{min}}$ and select from this group the section
   with the smallest weight per unit length. This is the most eco-
   nomical of the sections for which $S \geq S_{\text{min}}$. Note that this is not
   necessarily the section with the smallest value of $S$ (see Example
   5.04). In some cases, the selection of a section may be limited by
   other considerations, such as the allowable depth of the cross
   section, or the allowable deflection of the beam (cf. Chap. 9).

The foregoing discussion was limited to materials for which $\sigma_{\text{all}}$ is
the same in tension and in compression. If $\sigma_{\text{all}}$ is different in tension
and in compression, you should make sure to select the beam section
in such a way that $\sigma_m \leq \sigma_{\text{all}}$ for both tensile and compressive stresses.
If the cross section is not symmetric about its neutral axis, the largest
tensile and the largest compressive stresses will not necessarily occur in
the section where $|M|$ is maximum. One may occur where $M$ is maxi-
mum and the other where $M$ is minimum. Thus, step 2 should include
the determination of both $M_{\text{max}}$ and $M_{\text{min}}$, and step 3 should be modified
to take into account both tensile and compressive stresses.

Finally, keep in mind that the design procedure described in
this section takes into account only the normal stresses occurring on
the surface of the beam. Short beams, especially those made of tim-
ber, may fail in shear under a transverse loading. The determination
of shearing stresses in beams will be discussed in Chap. 6. Also, in
the case of rolled-steel beams, normal stresses larger than those con-
sidered here may occur at the junction of the web with the flanges.
This will be discussed in Chap. 8.

†We assume that all beams considered in this chapter are adequately braced to prevent
lateral buckling, and that bearing plates are provided under concentrated loads applied to
rolled-steel beams to prevent local buckling (crippling) of the web.
Load and Resistance Factor Design. This alternative method of design was briefly described in Sec. 1.13 and applied to members under axial loading. It can readily be applied to the design of beams in bending. Replacing in Eq. (1.26) the loads $P_D$, $P_L$, and $P_U$, respectively, by the bending moments $M_D$, $M_L$, and $M_U$, we write

$$\gamma_D M_D + \gamma_L M_L \leq \phi M_U$$

The coefficients $\gamma_D$ and $\gamma_L$ are referred to as the load factors and the coefficient $\phi$ as the resistance factor. The moments $M_D$ and $M_L$ are the bending moments due, respectively, to the dead and the live loads, while $M_U$ is equal to the product of the ultimate strength $\sigma_U$ of the material and the section modulus $S$ of the beam: $M_U = S\sigma_U$.

EXAMPLE 5.04

Select a wide-flange beam to support the 15-kip load as shown in Fig. 5.14. The allowable normal stress for the steel used is 24 ksi.

1. The allowable normal stress is given: $\sigma_{all} = 24$ ksi.
2. The shear is constant and equal to 15 kips. The bending moment is maximum at B. We have

$$M_{\text{max}} = (15 \text{ kips})(8 \text{ ft}) = 120 \text{ kip} \cdot \text{ft} = 1440 \text{ kip} \cdot \text{in.}$$

3. The minimum allowable section modulus is

$$S_{\text{min}} = \frac{M_{\text{max}}}{\sigma_{\text{all}}} = \frac{1440 \text{ kip} \cdot \text{in.}}{24 \text{ ksi}} = 60.0 \text{ in}^3$$

4. Referring to the table of Properties of Rolled-Steel Shapes in Appendix C, we note that the shapes are arranged in groups of the same depth and that in each group they are listed in order of decreasing weight. We choose in each group the lightest beam having a section modulus $S = I/c$ at least as large as $S_{\text{min}}$ and record the results in the following table.

<table>
<thead>
<tr>
<th>Shape</th>
<th>$S$, in$^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>W21 × 44</td>
<td>81.6</td>
</tr>
<tr>
<td>W18 × 50</td>
<td>88.9</td>
</tr>
<tr>
<td>W16 × 40</td>
<td>64.7</td>
</tr>
<tr>
<td>W14 × 43</td>
<td>62.6</td>
</tr>
<tr>
<td>W12 × 50</td>
<td>64.2</td>
</tr>
<tr>
<td>W10 × 54</td>
<td>60.0</td>
</tr>
</tbody>
</table>

The most economical is the W16 × 40 shape since it weighs only 40 lb/ft, even though it has a larger section modulus than two of the other shapes. We also note that the total weight of the beam will be $(8 \text{ ft}) \times (40 \text{ lb}) = 320 \text{ lb}$. This weight is small compared to the 15,000-lb load and can be neglected in our analysis.
SAMPLE PROBLEM 5.7

A 12-ft-long overhanging timber beam $AC$ with an 8-ft span $AB$ is to be designed to support the distributed and concentrated loads shown. Knowing that timber of 4-in. nominal width (3.5-in. actual width) with a 1.75-ksi allowable stress is to be used, determine the minimum required depth $h$ of the beam.

**SOLUTION**

**Reactions.** Considering the entire beam as a free body, we write

\[ + \sum M_A = 0: B(8 \text{ ft}) - (3.2 \text{ kips})(4 \text{ ft}) - (4.5 \text{ kips})(12 \text{ ft}) = 0 \]

\[ B = 8.35 \text{ kips} \quad \text{and} \quad A_x = 0 \]

\[ + \sum F_y = 0: A_y + 8.35 \text{ kips} - 3.2 \text{ kips} - 4.5 \text{ kips} = 0 \]

\[ A_y = -0.65 \text{ kips} \quad \text{and} \quad A_z = 0.65 \text{ kips} \]

**Shear Diagram.** The shear just to the right of $A$ is $V_A = A_x = -0.65$ kips. Since the change in shear between $A$ and $B$ is equal to minus the area under the load curve between these two points, we obtain $V_B$ by writing

\[ V_B - V_A = -(400 \text{ lb/ft})(8 \text{ ft}) = -3200 \text{ lb} = -3.20 \text{ kips} \]

\[ V_B = V_A - 3.20 \text{ kips} = -0.65 \text{ kips} - 3.20 \text{ kips} = -3.85 \text{ kips} \]

The reaction at $B$ produces a sudden increase of 8.35 kips in $V$, resulting in a value of the shear equal to 4.50 kips to the right of $B$. Since no load is applied between $B$ and $C$, the shear remains constant between these two points.

**Determination of $|M|_{\text{max}}$.** We first observe that the bending moment is equal to zero at both ends of the beam: $M_A = M_C = 0$. Between $A$ and $B$ the bending moment decreases by an amount equal to the area under the shear curve, and between $B$ and $C$ it increases by a corresponding amount. Thus, the maximum absolute value of the bending moment is $|M|_{\text{max}} = 18.00 \text{ kip \cdot ft}$.

**Minimum Allowable Section Modulus.** Substituting into Eq. (5.9) the given value of $\sigma_{\text{all}}$ and the value of $|M|_{\text{max}}$ that we have found, we write

\[ S_{\text{min}} = \frac{|M|_{\text{max}}}{\sigma_{\text{all}}} = \frac{(18 \text{ kip \cdot ft})(12 \text{ in./ft})}{1.75 \text{ ksi}} = 123.43 \text{ in}^3 \]

**Minimum Required Depth of Beam.** Recalling the formula developed in part 4 of the design procedure described in Sec. 5.4 and substituting the values of $b$ and $S_{\text{min}}$ we have

\[ h b h^2 \geq S_{\text{min}} \quad \text{and} \quad h(3.5 \text{ in.})h^2 \geq 123.43 \text{ in}^3 \quad h \geq 14.546 \text{ in.} \]

The minimum required depth of the beam is $h = 14.55$ in.

**Note:** In practice, standard wood shapes are specified by nominal dimensions that are slightly larger than actual. In this case, we would specify a 4-in. × 16-in. member, whose actual dimensions are 3.5 in. × 15.25 in.
SAMPLE PROBLEM 5.8

A 5-m-long, simply supported steel beam AD is to carry the distributed and concentrated loads shown. Knowing that the allowable normal stress for the grade of steel to be used is 160 MPa, select the wide-flange shape that should be used.

SOLUTION

Reactions. Considering the entire beam as a free body, we write

\[ +\sum F_x = 0: \quad A_x = 0 \]

\[ +\sum F_y = 0: \quad A_y = 52.0 \text{ kN} \quad A = 52.0 \text{ kN} \uparrow \]

Shear Diagram. The shear just to the right of A is \( V_A = A_y = +52.0 \text{ kN} \). Since the change in shear between A and B is equal to minus the area under the load curve between these two points, we have

\[ V_B = 52.0 \text{ kN} - 60 \text{ kN} = -8 \text{ kN} \]

The shear remains constant between B and C, where it drops to \(-58 \text{ kN}\), and keeps this value between C and D. We locate the section E of the beam where \( V = 0 \) by writing

\[ V_E - V_A = \text{tex} \]

\[ 0 - 52.0 \text{ kN} = -(20 \text{ kN/m})x \]

Solving for \( x \) we find \( x = 2.60 \text{ m} \).

Determination of \( |M|_{\text{max}} \). The bending moment is maximum at E, where \( V = 0 \). Since \( M \) is zero at the support A, its maximum value at E is equal to the area under the shear curve between A and E. We have, therefore, \( |M|_{\text{max}} = M_E = 67.6 \text{ kN} \cdot \text{m} \).

Minimum Allowable Section Modulus. Substituting into Eq. (5.9) the given value of \( \sigma_{\text{all}} \) and the value of \( |M|_{\text{max}} \) that we have found, we write

\[ S_{\text{min}} = \frac{|M|_{\text{max}}}{\sigma_{\text{all}}} = \frac{67.6 \text{ kN} \cdot \text{m}}{160 \text{ MPa}} = 422.5 \times 10^{-6} \text{ m}^3 = 422.5 \times 10^3 \text{ mm}^3 \]

Selection of Wide-Flange Shape. From Appendix C we compile a list of shapes that have a section modulus larger than \( S_{\text{min}} \) and are also the lightest shape in a given depth group.

<table>
<thead>
<tr>
<th>Shape</th>
<th>( S ) mm(^3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>W410 × 38.8</td>
<td>629</td>
</tr>
<tr>
<td>W360 × 32.9</td>
<td>475</td>
</tr>
<tr>
<td>W310 × 38.7</td>
<td>547</td>
</tr>
<tr>
<td>W250 × 44.8</td>
<td>531</td>
</tr>
<tr>
<td>W200 × 46.1</td>
<td>451</td>
</tr>
</tbody>
</table>

We select the lightest shape available, namely \( W360 \times 32.9 \).
5.65 and 5.66 For the beam and loading shown, design the cross section of the beam, knowing that the grade of timber used has an allowable normal stress of 12 MPa.

5.67 and 5.68 For the beam and loading shown, design the cross section of the beam, knowing that the grade of timber used has an allowable normal stress of 1750 psi.

5.69 and 5.70 For the beam and loading shown, design the cross section of the beam, knowing that the grade of timber used has an allowable normal stress of 12 MPa.
5.71 and 5.72 Knowing that the allowable stress for the steel used is 24 ksi, select the most economical wide-flange beam to support the loading shown.

Fig. P5.71

5.73 and 5.74 Knowing that the allowable stress for the steel used is 160 MPa, select the most economical wide-flange beam to support the loading shown.

Fig. P5.73

5.75 and 5.76 Knowing that the allowable stress for the steel used is 160 MPa, select the most economical S-shape beam to support the loading shown.

Fig. P5.75

5.77 and 5.78 Knowing that the allowable stress for the steel used is 24 ksi, select the most economical S-shape beam to support the loading shown.

Fig. P5.77
5.79 Two L102 × 76 rolled-steel angles are bolted together and used to support the loading shown. Knowing that the allowable normal stress for the steel used is 140 MPa, determine the minimum angle thickness that can be used.

Fig. P5.79

5.80 Two rolled-steel channels are to be welded back to back and used to support the loading shown. Knowing that the allowable normal stress for the steel used is 30 ksi, determine the most economical channels that can be used.

Fig. P5.80

5.81 Three steel plates are welded together to form the beam shown. Knowing that the allowable normal stress for the steel used is 22 ksi, determine the minimum flange width b that can be used.

Fig. P5.81

5.82 A steel pipe of 100-mm diameter is to support the loading shown. Knowing that the stock of pipes available has thicknesses varying from 6 mm to 24 mm in 3-mm increments, and that the allowable normal stress for the steel used is 150 MPa, determine the minimum wall thickness t that can be used.

Fig. P5.82

5.83 Assuming the upward reaction of the ground to be uniformly distributed and knowing that the allowable normal stress for the steel used is 24 ksi, select the most economical wide-flange beam to support the loading shown.

Fig. P5.83

5.84 Assuming the upward reaction of the ground to be uniformly distributed and knowing that the allowable normal stress for the steel used is 170 MPa, select the most economical wide-flange beam to support the loading shown.
5.85 and 5.86 Determine the largest permissible value of $P$ for the beam and loading shown, knowing that the allowable normal stress is $+6$ ksi in tension and $-18$ ksi in compression.

Fig. P5.85

5.87 Determine the largest permissible distributed load $w$ for the beam shown, knowing that the allowable normal stress is $+80$ MPa in tension and $-130$ MPa in compression.

Fig. P5.87

5.88 Solve Prob. 5.87, assuming that the cross section of the beam is reversed, with the flange of the beam resting on the supports at $B$ and $C$.

5.89 A 54-kip load is to be supported at the center of the 16-ft span shown. Knowing that the allowable normal stress for the steel used is 24 ksi, determine $(a)$ the smallest allowable length $l$ of beam $CD$ if the W12 × 50 beam $AB$ is not to be overstressed, $(b)$ the most economical W shape that can be used for beam $CD$. Neglect the weight of both beams.

Fig. P5.89

5.90 A uniformly distributed load of 66 kN/m is to be supported over the 6-m span shown. Knowing that the allowable normal stress for the steel used is 140 MPa, determine $(a)$ the smallest allowable length $l$ of beam $CD$ if the W460 × 74 beam $AB$ is not to be overstressed, $(b)$ the most economical W shape that can be used for beam $CD$. Neglect the weight of both beams.

Fig. P5.90
Each of the three rolled-steel beams shown (numbered 1, 2, and 3) is to carry a 64-kip load uniformly distributed over the beam. Each of these beams has a 12-ft span and is to be supported by the two 24-ft rolled-steel girders $AC$ and $BD$. Knowing that the allowable normal stress for the steel used is 24 ksi, select (a) the most economical S shape for the three beams, (b) the most economical W shape for the two girders.

Fig. P5.91

5.92 Beams $AB$, $BC$, and $CD$ have the cross section shown and are pin-connected at $B$ and $C$. Knowing that the allowable normal stress is $+110$ MPa in tension and $-150$ MPa in compression, determine (a) the largest permissible value of $w$ if beam $BC$ is not to be overstressed, (b) the corresponding maximum distance $a$ for which the cantilever beams $AB$ and $CD$ are not overstressed.

Fig. P5.92
5.93 Beams AB, BC, and CD have the cross section shown and are pin-connected at B and C. Knowing that the allowable normal stress is +110 MPa in tension and −150 MPa in compression, determine (a) the largest permissible value of $P$ if beam BC is not to be overstressed, (b) the corresponding maximum distance $a$ for which the cantilever beams AB and CD are not overstressed.

![Fig. P5.93](image)

*5.94* A bridge of length $L = 48$ ft is to be built on a secondary road whose access to trucks is limited to two-axle vehicles of medium weight. It will consist of a concrete slab and of simply supported steel beams with an ultimate strength $s_U = 60$ ksi. The combined weight of the slab and beams can be approximated by a uniformly distributed load $w = 0.75$ kips/ft on each beam. For the purpose of the design, it is assumed that a truck with axles located at a distance $a = 14$ ft from each other will be driven across the bridge and that the resulting concentrated loads $P_1$ and $P_2$ exerted on each beam could be as large as 24 kips and 6 kips, respectively. Determine the most economical wide-flange shape for the beams, using LRFD with the load factors $g_D = 1.25$, $g_L = 1.75$ and the resistance factor $\phi = 0.9$. [Hint: It can be shown that the maximum value of $|M_{L1}|$ occurs under the larger load when that load is located to the left of the center of the beam at a distance equal to $aP_2/(P_1 + P_2)$.]
5.5 Using Singularity Functions to Determine Shear and Bending Moment in a Beam

Reviewing the work done in the preceding sections, we note that the shear and bending moment could only rarely be described by single analytical functions. In the case of the cantilever beam of Example 5.02 (Fig. 5.9), which supported a uniformly distributed load \( w \), the shear and bending moment could be represented by single analytical functions, namely, \( V = -wx \) and \( M = -\frac{1}{2}wx^2 \); this was due to the fact that no discontinuity existed in the loading of the beam. On the other hand, in the case of the simply supported beam of Example 5.01, which was loaded only at its midpoint \( C \), the load \( P \) applied at \( C \) represented a singularity in the beam loading. This singularity resulted in discontinuities in the shear and bending moment and required the use of different analytical functions to represent \( V \) and \( M \) in the portions of beam located, respectively, to the left and to the right of point \( C \). In Sample Prob. 5.2, the beam had to be divided into three portions, in each of which different functions were used to represent the shear and the bending moment. This situation led us to rely on the graphical representation of the functions \( V \) and \( M \) provided by the shear and bending-moment diagrams and, later in Sec. 5.3, on a graphical method of integration to determine \( V \) and \( M \) from the distributed load \( w \).

The purpose of this section is to show how the use of singularity functions makes it possible to represent the shear \( V \) and the bending moment \( M \) by single mathematical expressions.

Consider the simply supported beam \( AB \), of length \( 2a \), which carries a uniformly distributed load \( w_0 \) extending from its midpoint \( C \) to its right-hand support \( B \) (Fig. 5.15). We first draw the free-body diagram of the entire beam (Fig. 5.16a); replacing the distributed load by an equivalent concentrated load and, summing moments about \( B \), we write

\[
+\gamma \Sigma M_B = 0: \quad (w_0a)\left(\frac{1}{2}a\right) - R_A(2a) = 0 \quad R_A = \frac{1}{2}w_0a
\]

Next we cut the beam at a point \( D \) between \( A \) and \( C \). From the free-body diagram of \( AD \) (Fig. 5.16b) we conclude that, over the interval \( 0 < x < a \), the shear and bending moment are expressed, respectively, by the functions

\[
V_1(x) = \frac{1}{2}w_0a \quad \text{and} \quad M_1(x) = \frac{1}{4}w_0ax
\]

Cutting now, the beam at a point \( E \) between \( C \) and \( B \), we draw the free-body diagram of portion \( AE \) (Fig. 5.16c). Replacing the distributed load by an equivalent concentrated load, we write

\[
+\gamma \Sigma F_y = 0: \quad \frac{1}{2}w_0a + w_0(x-a) - V_2 = 0
\]

\[
+\gamma \Sigma M_E = 0: \quad -\frac{1}{2}w_0ax + w_0(x-a) \left[\frac{1}{2}(x-a)\right] + M_2 = 0
\]

and conclude that, over the interval \( a < x < 2a \), the shear and bending moment are expressed, respectively, by the functions

\[
V_2(x) = \frac{1}{2}w_0a - w_0(x-a) \quad \text{and} \quad M_2(x) = \frac{1}{4}w_0ax - \frac{1}{2}w_0(x-a)^2
\]
As we pointed out earlier in this section, the fact that the shear and bending moment are represented by different functions of \( x \), depending upon whether \( x \) is smaller or larger than \( a \), is due to the discontinuity in the loading of the beam. However, the functions \( V_1(x) \) and \( V_2(x) \) can be represented by the single expression

\[
V(x) = \frac{1}{4}w_0 a - \frac{1}{2}w_0(x - a)
\]  

(5.11)

if we specify that the second term should be included in our computations when \( x \geq a \) and ignored when \( x < a \). In other words, the brackets \( \langle \rangle \) should be replaced by ordinary parentheses \((\)\) when \( x \geq a \) and by zero when \( x < a \). With the same convention, the bending moment can be represented at any point of the beam by the single expression

\[
M(x) = \frac{1}{4}w_0 d x - \frac{1}{2}w_0(x - a)^2
\]  

(5.12)

From the convention we have adopted, it follows that brackets \( \langle \rangle \) can be differentiated or integrated as ordinary parentheses. Instead of calculating the bending moment from free-body diagrams, we could have used the method indicated in Sec. 5.3 and integrated the expression obtained for \( V(x) \):

\[
M(x) - M(0) = \int_0^x V(x) \, dx = \int_0^x \frac{1}{4}w_0 a \, dx - \int_0^x w_0(x - a) \, dx
\]

After integration, and observing that \( M(0) = 0 \), we obtain as before

\[
M(x) = \frac{1}{4}w_0 a - \frac{1}{2}w_0(x - a)^2
\]  

Furthermore, using the same convention again, we note that the distributed load at any point of the beam can be expressed as

\[
w(x) = w_0(x - a)^0
\]  

(5.13)

Indeed, the brackets should be replaced by zero for \( x < a \) and by parentheses for \( x \geq a \); we thus check that \( w(x) = 0 \) for \( x < a \) and, defining the zero power of any number as unity, that \( \langle x - a \rangle^0 = (x - a)^0 = 1 \) and \( w(x) = w_0 \) for \( x \geq a \). From Sec. 5.3 we recall that the shear could have been obtained by integrating the function \( -w(x) \). Observing that \( V = \frac{1}{4}w_0 a \) for \( x = 0 \), we write

\[
V(x) - V(0) = -\int_0^x w(x) \, dx = -\int_0^x w_0(x - a)^0 \, dx
\]

\[
V(x) - \frac{1}{4}w_0 a = -w_0(x - a)^1
\]

Solving for \( V(x) \) and dropping the exponent 1, we obtain again

\[
V(x) = \frac{1}{4}w_0 a - w_0(x - a)
\]

The expressions \( \langle x - a \rangle^0, \langle x - a \rangle, \langle x - a \rangle^2 \) are called singularity functions. By definition, we have, for \( n \geq 0 \),

\[
\langle x - a \rangle^n = \begin{cases}
(x - a)^n & \text{when } x \geq a \\
0 & \text{when } x < a
\end{cases}
\]  

(5.14)
We also note that whenever the quantity between brackets is positive or zero, the brackets should be replaced by ordinary parentheses, and whenever that quantity is negative, the bracket itself is equal to zero.

\[
\langle x-a \rangle^0 = \begin{cases} 
1 & \text{when } x \geq a \\
0 & \text{when } x < a 
\end{cases} 
\]  
(5.15)

It follows from the definition of singularity functions that

\[
\int \langle x-a \rangle^n \, dx = \frac{1}{n+1} \langle x-a \rangle^{n+1} \quad \text{for } n \geq 0
\]  
(5.16)

and

\[
\frac{d}{dx} \langle x-a \rangle^n = n \langle x-a \rangle^{n-1} \quad \text{for } n \geq 1
\]  
(5.17)

Most of the beam loadings encountered in engineering practice can be broken down into the basic loadings shown in Fig. 5.18. Whenever applicable, the corresponding functions \(w(x), V(x)\), and \(M(x)\) have been expressed in terms of singularity functions and plotted against a color background. A heavier color background was used to indicate for each loading the expression that is most easily derived or remembered and from which the other functions can be obtained by integration.

†Since \((x-a)^0\) is discontinuous at \(x = a\), it can be argued that this function should be left undefined for \(x = a\) or that it should be assigned both of the values 0 and 1 for \(x = a\). However, defining \((x-a)^0\) as equal to 1 when \(x = a\), as stated in (Eq. 5.15), has the advantage of being unambiguous and, thus, readily applicable to computer programming (cf. page 348).
After a given beam loading has been broken down into the basic loadings of Fig. 5.18, the functions $V(x)$ and $M(x)$ representing the shear and bending moment at any point of the beam can be obtained by adding the corresponding functions associated with each of the basic loadings and reactions. Since all the distributed loadings shown in Fig. 5.18 are open-ended to the right, a distributed loading
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that does not extend to the right end of the beam or that is discontinuous should be replaced as shown in Fig. 5.19 by an equivalent combination of open-ended loadings. (See also Example 5.05 and Sample Prob. 5.9.)

As you will see in Sec. 9.6, the use of singularity functions also greatly simplifies the determination of beam deflections. It was in connection with that problem that the approach used in this section was first suggested in 1862 by the German mathematician A. Clebsch (1833–1872). However, the British mathematician and engineer W. H. Macaulay (1853–1936) is usually given credit for introducing the singularity functions in the form used here, and the brackets \( \langle \rangle \) are generally referred to as Macaulay’s brackets.†


For the beam and loading shown (Fig. 5.20a) and using singularity functions, express the shear and bending moment as functions of the distance \( x \) from the support at \( A \).

We first determine the reaction at \( A \) by drawing the free-body diagram of the beam (Fig. 5.20b) and writing

\[
\begin{align*}
\sum F_x &= 0: \quad A_x = 0 \\
\sum M_A &= 0: \quad -A_y(3.6 \text{ m}) + (1.2 \text{ kN})(3 \text{ m}) \\
&+ (1.8 \text{ kN})(2.4 \text{ m}) + 1.44 \text{ kN} \cdot \text{m} = 0 \\
A_y &= 2.60 \text{ kN}
\end{align*}
\]

Next, we replace the given distributed loading by two equivalent open-ended loadings (Fig. 5.20c) and express the distributed load \( w(x) \) as the sum of the corresponding step functions:

\[
w(x) = +w_0(x - 0.6)^0 - w_0(x - 1.8)^0
\]

The function \( V(x) \) is obtained by integrating \( w(x) \), reversing the + and − signs, and adding to the result the constants \( A_y \) and \( -P(x - 0.6)^0 \) representing the respective contributions to the shear of the reaction at \( A \) and of the concentrated load. (No other constant of integration is required.) Since the concentrated couple does not directly affect the shear, it should be ignored in this computation. We write

\[
V(x) = -w_0(x - 0.6)^1 + w_0(x - 1.8)^1 + A_y - P(x - 0.6)^0
\]

In a similar way, the function \( M(x) \) is obtained by integrating \( V(x) \) and adding to the result the constant \( -M_0(x - 2.6)^0 \) representing the contribution of the concentrated couple to the bending moment. We have

\[
M(x) = -\frac{1}{2}w_0(x - 0.6)^2 + \frac{1}{2}w_0(x - 1.8)^2 + A_y - P(x - 0.6)^1 - M_0(x - 2.6)^0
\]
Substituting the numerical values of the reaction and loads into the expressions obtained for $V(x)$ and $M(x)$ and being careful not to compute any product or expand any square involving a bracket, we obtain the following expressions for the shear and bending moment at any point of the beam:

\[
V(x) = -1.5(x - 0.6)^1 + 1.5(x - 1.8)^1 + 2.6 - 1.2(x - 0.6)^0
\]

\[
M(x) = -0.75(x - 0.6)^2 + 0.75(x - 1.8)^2 + 2.6x - 1.2(x - 0.6)^1 - 1.44(x - 2.6)^0
\]

**EXAMPLE 5.06**

For the beam and loading of Example 5.05, determine the numerical values of the shear and bending moment at the midpoint $D$.

Making $x = 1.8$ m in the expressions found for $V(x)$ and $M(x)$ in Example 5.05, we obtain

\[
V(1.8) = -1.5(1.2)^1 + 1.5(0)^1 + 2.6 - 1.2(1.2)^0
\]

\[
M(1.8) = -0.75(1.2)^2 + 0.75(0)^2 + 2.6(1.8) - 1.2(1.2)^1 - 1.44(-0.8)^0
\]

Recalling that whenever a quantity between brackets is positive or zero, the brackets should be replaced by ordinary parentheses, and whenever the quantity is negative, the bracket itself is equal to zero, we write

\[
V(1.8) = -1.5(1.2) + 1.5(0) + 2.6 - 1.2(1)
\]

\[
= -1.5 + 0 + 2.6 - 1.2
\]

\[
V(1.8) = -0.4 \text{ kN}
\]

and

\[
M(1.8) = -0.75(1.2)^2 + 0.75(0)^2 + 2.6(1.8) - 1.2(1.2)^1 - 1.44(0)
\]

\[
= -1.08 + 0 + 4.68 - 1.44 - 0
\]

\[
M(1.8) = +2.16 \text{ kN} \cdot \text{m}
\]
**SAMPLE PROBLEM 5.9**

For the beam and loading shown, determine (a) the equations defining the shear and bending moment at any point, (b) the shear and bending moment at points C, D, and E.

**SOLUTION**

**Reactions.** The total load is \( \frac{1}{2} w_0 L \); because of symmetry, each reaction is equal to half that value, namely, \( \frac{1}{4} w_0 L \).

**Distributed Load.** The given distributed loading is replaced by two equivalent open-ended loadings as shown. Using a singularity function to express the second loading, we write

\[
W(x) = k_1 x + k_2 (x - \frac{L}{2}) = \frac{2w_0}{L} x - \frac{4w_0}{L} (x - \frac{L}{4})
\]  

(1)

\( k_1 = \frac{2w_0}{L} \)

\( k_2 = -\frac{4w_0}{L} \)

We obtain \( V(x) \) by integrating (1), changing the signs, and adding a constant equal to \( R_A \):

\[
V(x) = -\frac{w_0}{3} x^3 + \frac{2w_0}{L} (x - \frac{L}{4})^2 + \frac{1}{3} w_0 L 
\]  

(2)

We obtain \( M(x) \) by integrating (2); since there is no concentrated couple, no constant of integration is needed:

\[
M(x) = -\frac{w_0}{3} x^3 + \frac{2w_0}{3L} (x - \frac{L}{4})^3 + \frac{1}{12} w_0 L x 
\]  

(3)

**a. Equations for Shear and Bending Moment.** We obtain \( V(x) \) by integrating (1), changing the signs, and adding a constant equal to \( R_A \):

\[
V(x) = -\frac{w_0}{3} x^3 + \frac{2w_0}{L} (x - \frac{L}{4})^2 + \frac{1}{3} w_0 L 
\]  

(2)

We obtain \( M(x) \) by integrating (2); since there is no concentrated couple, no constant of integration is needed:

\[
M(x) = -\frac{w_0}{3} x^3 + \frac{2w_0}{3L} (x - \frac{L}{4})^3 + \frac{1}{12} w_0 L x 
\]  

(3)

**b. Shear and Bending Moment at C, D, and E**

**At Point C:** Making \( x = \frac{L}{2} \) in Eqs. (2) and (3) and recalling that whenever a quantity between brackets is positive or zero, the brackets may be replaced by parentheses, we have

\[
V_C = -\frac{w_0}{3} \left(\frac{L}{2}\right)^3 + \frac{2w_0}{L} \left(\frac{L}{4}\right)^2 + \frac{1}{3} w_0 L 
\]  

\( V_C = 0 \)

\[
M_C = -\frac{w_0}{3} \left(\frac{L}{2}\right)^3 + \frac{2w_0}{3L} \left(\frac{L}{4}\right)^3 + \frac{1}{12} w_0 L \left(\frac{L}{4}\right) 
\]  

\( M_C = \frac{1}{12} w_0 L^2 \)

**At Point D:** Making \( x = \frac{L}{3} \) in Eqs. (2) and (3) and recalling that a bracket containing a negative quantity is equal to zero, we write

\[
V_D = -\frac{w_0}{3} \left(\frac{L}{3}\right)^3 + \frac{2w_0}{L} \left(\frac{1}{3} L\right)^2 + \frac{1}{3} w_0 L 
\]  

\( V_D = \frac{3}{16} w_0 L \)

\[
M_D = -\frac{w_0}{3} \left(\frac{L}{3}\right)^3 + \frac{2w_0}{3L} \left(\frac{1}{3} L\right)^3 + \frac{1}{12} w_0 L \left(\frac{1}{3} L\right) 
\]  

\( M_D = \frac{11}{192} w_0 L^2 \)

**At Point E:** Making \( x = \frac{5}{2} L \) in Eqs. (2) and (3), we have

\[
V_E = -\frac{w_0}{3} \left(\frac{5}{2} L\right)^3 + \frac{2w_0}{L} \left(\frac{5}{2} L\right)^2 + \frac{1}{3} w_0 L 
\]  

\( V_E = -\frac{3}{16} w_0 L \)

\[
M_E = -\frac{w_0}{3} \left(\frac{5}{2} L\right)^3 + \frac{2w_0}{3L} \left(\frac{5}{2} L\right)^3 + \frac{1}{12} w_0 L \left(\frac{5}{2} L\right) 
\]  

\( M_E = \frac{11}{192} w_0 L^2 \)
SAMPLE PROBLEM 5.10

The rigid bar DEF is welded at point D to the steel beam AB. For the loading shown, determine (a) the equations defining the shear and bending moment at any point of the beam, (b) the location and magnitude of the largest bending moment.

SOLUTION

Reactions. We consider the beam and bar as a free body and observe that the total load is 960 lb. Because of symmetry, each reaction is equal to 480 lb.

Modified Loading Diagram. We replace the 160-lb load applied at F by an equivalent force-couple system at D. We thus obtain a loading diagram consisting of a concentrated couple, three concentrated loads (including the two reactions), and a uniformly distributed load

$$w(x) = 50 \text{ lb/ft}$$

(a) Equations for Shear and Bending Moment. We obtain $V(x)$ by integrating (1), changing the sign, and adding constants representing the respective contributions of $R_A$ and $P$ to the shear. Since $P$ affects $V(x)$ only for values of $x$ larger than 11 ft, we use a step function to express its contribution.

$$V(x) = -50x + 480 - 160(x - 11)^0$$

We obtain $M(x)$ by integrating (2) and using a step function to represent the contribution of the concentrated couple $M_D$:

$$M(x) = -25x^2 + 480x - 160(x - 11)^1 - 480(x - 11)^0$$

(b) Largest Bending Moment. Since $M$ is maximum or minimum when $V = 0$, we set $V = 0$ in (2) and solve that equation for $x$ to find the location of the largest bending moment. Considering first values of $x$ less than 11 ft and noting that for such values the bracket is equal to zero, we write

$$-50x + 480 = 0 \quad x = 9.60 \text{ ft}$$

Considering now values of $x$ larger than 11 ft, for which the bracket is equal to 1, we have

$$-50x + 480 - 160 = 0 \quad x = 6.40 \text{ ft}$$

Since this value is not larger than 11 ft, it must be rejected. Thus, the value of $x$ corresponding to the largest bending moment is

$$x_m = 9.60 \text{ ft}$$

Substituting this value for $x$ into Eq. (3), we obtain

$$M_{\text{max}} = -25(9.60)^2 + 480(9.60) - 160(-1.40)^1 - 480(-1.40)^0$$

and, recalling that brackets containing a negative quantity are equal to zero,

$$M_{\text{max}} = -25(9.60)^2 + 480(9.60)$$

$M_{\text{max}} = 2304 \text{ lb} \cdot \text{ft}$

The bending-moment diagram has been plotted. Note the discontinuity at point D due to the concentrated couple applied at that point. The values of $M$ just to the left and just to the right of $D$ were obtained by making $x = 11$ in Eq. (3) and replacing the step function $(x - 11)^0$ by 0 and 1, respectively.
5.98 through 5.100  (a) Using singularity functions, write the equations defining the shear and bending moment for the beam and loading shown.  (b) Use the equation obtained for $M$ to determine the bending moment at point $C$ and check your answer by drawing the free-body diagram of the entire beam.

5.101 through 5.103  (a) Using singularity functions, write the equations defining the shear and bending moment for the beam and loading shown.  (b) Use the equation obtained for $M$ to determine the bending moment at point $E$ and check your answer by drawing the free-body diagram of the portion of the beam to the right of $E$.

5.104  (a) Using singularity functions, write the equations for the shear and bending moment for beam $ABC$ under the loading shown.  (b) Use the equation obtained for $M$ to determine the bending moment just to the right of point $B$.

5.105  (a) Using singularity functions, write the equations for the shear and bending moment for beam $ABC$ under the loading shown.  (b) Use the equation obtained for $M$ to determine the bending moment just to the right of point $D$. 
5.106 through 5.109 (a) Using singularity functions, write the equations for the shear and bending moment for the beam and loading shown. (b) Determine the maximum value of the bending moment in the beam.

![Fig. P5.106](image)

![Fig. P5.107](image)

5.110 and 5.111 (a) Using singularity functions, write the equations for the shear and bending moment for the beam and loading shown. (b) Determine the maximum normal stress due to bending.

![Fig. P5.110](image)

![Fig. P5.111](image)

5.112 and 5.113 (a) Using singularity functions, find the magnitude and location of the maximum bending moment for the beam and loading shown. (b) Determine the maximum normal stress due to bending.

![Fig. P5.112](image)

![Fig. P5.113](image)
5.114 and 5.115 A beam is being designed to be supported and loaded as shown. (a) Using singularity functions, find the magnitude and location of the maximum bending moment in the beam. (b) Knowing that the allowable normal stress for the steel to be used is 24 ksi, find the most economical wide-flange shape that can be used.

![Fig. P5.114](image1)

![Fig. P5.115](image2)

5.116 and 5.117 A timber beam is being designed with supports and loads as shown. (a) Using singularity functions, find the magnitude and location of the maximum bending moment in the beam. (b) Knowing that the available stock consists of beams with an allowable stress of 12 MPa and a rectangular cross section of 30-mm width and depth \( h \) varying from 80 mm to 160 mm in 10-mm increments, determine the most economical cross section that can be used.

![Fig. P5.116](image3)

![Fig. P5.117](image4)

5.118 through 5.121 Using a computer and step functions, calculate the shear and bending moment for the beam and loading shown. Use the specified increment \( \Delta L \), starting at point A and ending at the right-hand support.

![Fig. P5.118](image5)

![Fig. P5.119](image6)

![Fig. P5.120](image7)

![Fig. P5.121](image8)
5.122 and 5.123 For the beam and loading shown, and using a computer and step functions, (a) tabulate the shear, bending moment, and maximum normal stress in sections of the beam from \( x = 0 \) to \( x = L \), using the increments \( \Delta L \) indicated, (b) using smaller increments if necessary, determine with a 2\% accuracy the maximum normal stress in the beam. Place the origin of the \( x \) axis at end \( A \) of the beam.

![Fig. P5.122](image1)

5.124 and 5.125 For the beam and loading shown, and using a computer and step functions, (a) tabulate the shear, bending moment, and maximum normal stress in sections of the beam from \( x = 0 \) to \( x = L \), using the increments \( \Delta L \) indicated, (b) using smaller increments if necessary, determine with a 2\% accuracy the maximum normal stress in the beam. Place the origin of the \( x \) axis at end \( A \) of the beam.

![Fig. P5.123](image2)

![Fig. P5.124](image3)

![Fig. P5.125](image4)

*5.6 NONPRISOMATIC BEAMS*

Our analysis has been limited so far to prismatic beams, i.e., to beams of uniform cross section. As we saw in Sec. 5.4, prismatic beams are designed so that the normal stresses in their critical sections are at most equal to the allowable value of the normal stress for the material being used. It follows that, in all other sections, the normal stresses will be smaller, possibly much smaller, than their allowable value. A prismatic beam, therefore, is almost always overdesigned, and considerable savings of material can be realized by using nonprismatic beams, i.e., beams of variable cross section. The cantilever beams shown in the bridge during construction in Photo 5.2 are examples of nonprismatic beams.

Since the maximum normal stresses \( \sigma_m \) usually control the design of a beam, the design of a nonprismatic beam will be optimum if the
section modulus $S = I/c$ of every cross section satisfies Eq. (5.3) of Sec. 5.1. Solving that equation for $S$, we write

$$S = \frac{|M|}{\sigma_{all}} \tag{5.18}$$

A beam designed in this manner is referred to as a beam of constant strength.

For a forged or cast structural or machine component, it is possible to vary the cross section of the component along its length and to eliminate most of the unnecessary material (see Example 5.07). For a timber beam or a rolled-steel beam, however, it is not possible to vary the cross section of the beam. But considerable savings of material can be achieved by gluing wooden planks of appropriate lengths to a timber beam (see Sample Prob. 5.11) and using cover plates in portions of a rolled-steel beam where the bending moment is large (see Sample Prob. 5.12).

**EXAMPLE 5.07**

A cast-aluminum plate of uniform thickness $b$ is to support a uniformly distributed load $w$ as shown in Fig. 5.21. (a) Determine the shape of the plate that will yield the most economical design. (b) Knowing that the allowable normal stress for the aluminum used is 72 MPa and that $b = 40$ mm, $L = 800$ mm, and $w = 135$ kN/m, determine the maximum depth $h_0$ of the plate.

**Bending Moment.**

Measuring the distance $x$ from $A$ and observing that $V_A = M_A = 0$, we use Eqs. (5.6) and (5.8) of Sec. 5.3 and write

$$V(x) = -\int_0^x wdx = -wx$$

$$M(x) = \int_0^x V(x)dx = -\int_0^x wdx = -\frac{1}{2}wx^2$$

(a) **Shape of Plate.** We recall from Sec. 5.4 that the modulus $S$ of a rectangular cross section of width $b$ and depth $h$ is $S = \frac{b}{2}bh^2$. Carrying this value into Eq. (5.18) and solving for $h^2$, we have

$$h^2 = \frac{6|M|}{b\sigma_{all}} \tag{5.19}$$

and, after substituting $|M| = \frac{1}{2}wx^2$,

$$h^2 = \frac{3wx^2}{b\sigma_{all}} \text{ or } h = \left(\frac{3w}{b\sigma_{all}}\right)^{1/2}x \tag{5.20}$$

Since the relation between $h$ and $x$ is linear, the lower edge of the plate is a straight line. Thus, the plate providing the most economical design is of triangular shape.

(b) **Maximum Depth $h_0$.** Making $x = L$ in Eq. (5.20) and substituting the given data, we obtain

$$h_0 = \left[\frac{3(135 \text{ kN/m})}{(0.040 \text{ m})(72 \text{ MPa})}\right]^{1/2} (800 \text{ mm}) = 300 \text{ mm}$$
**SAMPLE PROBLEM 5.11**

A 12-ft-long beam made of a timber with an allowable normal stress of 2.40 ksi and an allowable shearing stress of 0.40 ksi is to carry two 4.8-kip loads located at its third points. As shown in Chap. 6, a beam of uniform rectangular cross section, 4 in. wide and 4.5 in. deep, would satisfy the allowable shearing stress requirement. Since such a beam would not satisfy the allowable normal stress requirement, it will be reinforced by gluing planks of the same timber, 4 in. wide and 1.25 in. thick, to the top and bottom of the beam in a symmetric manner. Determine (a) the required number of pairs of planks, (b) the length of the planks in each pair that will yield the most economical design.

**SOLUTION**

**Bending Moment.** We draw the free-body diagram of the beam and find the following expressions for the bending moment:

From A to B (0 ≤ x ≤ 48 in.):

\[ M = (4.80 \text{ kips})x \]

From B to C (48 in. ≤ x ≤ 96 in.):

\[ M = (4.80 \text{ kips})x - (4.80 \text{ kips})(x - 48 \text{ in.}) = 230.4 \text{ kip \cdot in.} \]

**a. Number of Pairs of Planks.** We first determine the required total depth of the reinforced beam between B and C. We recall from Sec. 5.4 that

\[ S = \frac{6bh^2}{M} \]

for a beam with a rectangular cross section of width b and depth h. Substituting this value into Eq. (5.17) and solving for \( h^2 \), we have

\[ h^2 = \frac{6|M|}{b\sigma_{all}} \]  

(1)

Substituting the value obtained for \( M \) from B to C and the given values of \( b \) and \( \sigma_{all} \), we write

\[ h^2 = \frac{6(230.4 \text{ kip \cdot in.})}{(4 \text{ in.})(2.40 \text{ ksi})} = 144 \text{ in.}^2 \quad h = 12.00 \text{ in.} \]

Since the original beam has a depth of 4.50 in., the planks must provide an additional depth of 7.50 in. Recalling that each pair of planks is 2.50 in. thick, we write:

Required number of pairs of planks = 3

**b. Length of Planks.** The bending moment was found to be \( M = (4.80 \text{ kips})x \) in the portion AB of the beam. Substituting this expression and the given values of \( b \) and \( \sigma_{all} \) into Eq. (1) and solving for \( x \), we have

\[ x = \frac{(4 \text{ in.})(2.40 \text{ ksi})}{6(4.80 \text{ kips})}h^2 = \frac{h^2}{3 \text{ in.}} \]  

(2)

Equation (2) defines the maximum distance \( x \) from end A at which a given depth \( h \) of the cross section is acceptable. Making \( h = 4.50 \text{ in.} \), we find the distance \( x_1 \) from A at which the original prismatic beam is safe: \( x_1 = 6.75 \text{ in.} \). From that point on, the original beam should be reinforced by the first pair of planks. Making \( h = 4.50 \text{ in.} + 2.50 \text{ in.} = 7.00 \text{ in.} \), yields the distance \( x_2 = 16.33 \text{ in.} \) from which the second pair of planks should be used, and making \( h = 9.50 \text{ in.} \), yields the distance \( x_3 = 30.08 \text{ in.} \) from which the third pair of planks should be used. The length \( l_i \) of the planks of the pair \( i \), where \( i = 1, 2, 3 \), is obtained by subtracting \( 2x_i \) from the 144-in. length of the beam. We find

\[ l_1 = 130.5 \text{ in.}, l_2 = 111.3 \text{ in.}, l_3 = 83.8 \text{ in.} \]

The corners of the various planks lie on the parabola defined by Eq. (2).
SAMPLE PROBLEM 5.12

Two steel plates, each 16 mm thick, are welded as shown to a W690 × 125 beam to reinforce it. Knowing that σ_{all} = 160 MPa for both the beam and the plates, determine the required value of (a) the length of the plates, (b) the width of the plates.

SOLUTION

Bending Moment. We first find the reactions. From the free body of a portion of beam of length x ≤ 4 m, we obtain M between A and C:

\[ M = (250 \text{ kN})x \]  \hspace{1cm} (1)

a. Required Length of Plates. We first determine the maximum allowable length \( x_m \) of the portion AD of the unreinforced beam. From Appendix C we find that the section modulus of a W690 × 125 beam is

\[ S = 3490 \times 10^6 \text{ mm}^3, \]  

or

\[ S = 3.49 \times 10^3 \text{ m}^3. \]

Substituting for \( S \) and \( \sigma_{all} \) into Eq. (5.17) and solving for M, we write

\[ M = S\sigma_{all} = (3.49 \times 10^3 \text{ m}^3)(160 \times 10^6 \text{ kN/m}^2) = 558.4 \text{ kN} \cdot \text{m} \]

Substituting for M in Eq. (1), we have

\[ 558.4 \text{ kN} \cdot \text{m} = (250 \text{ kN})x_m \]  

\[ x_m = 2.234 \text{ m} \]

The required length \( l \) of the plates is obtained by subtracting 2\( x_m \) from the length of the beam:

\[ l = 8 \text{ m} - 2(2.234 \text{ m}) = 3.532 \text{ m} \]

\[ l = 3.53 \text{ m} \]

b. Required Width of Plates. The maximum bending moment occurs in the midsection C of the beam. Making \( x = 4 \text{ m} \) in Eq. (1), we obtain the bending moment in that section:

\[ M = (250 \text{ kN})(4 \text{ m}) = 1000 \text{ kN} \cdot \text{m} \]

In order to use Eq. (5.1) of Sec. 5.1, we now determine the moment of inertia of the cross section of the reinforced beam with respect to a centroidal axis and the distance \( c \) from that axis to the outer surfaces of the plates. From Appendix C we find that the moment of inertia of a W690 × 125 beam is \( I_b = 1190 \times 10^6 \text{ mm}^4 \) and its depth is \( d = 678 \text{ mm} \). On the other hand, denoting by \( t \) the thickness of one plate, by \( b \) its width, and by \( \overline{y} \) the distance of its centroid from the neutral axis, we express the moment of inertia \( I_p \) of the two plates with respect to the neutral axis:

\[ I_p = 2b(t^3 + A\overline{y}^2) = \left(\frac{1}{6}t^3\right)b + 2bt\left(\frac{1}{2}d + \frac{1}{2}t\right)^2 \]

Substituting \( t = 16 \text{ mm} \) and \( d = 678 \text{ mm} \), we obtain \( I_p = (3.854 \times 10^6 \text{ mm}^4)b \). The moment of inertia \( I \) of the beam and plates is

\[ I = I_b + I_p = 1190 \times 10^6 \text{ mm}^4 + (3.854 \times 10^6 \text{ mm}^4)b \]  \hspace{1cm} (2)

and the distance from the neutral axis to the surface is \( c = \frac{1}{2}d + t = 355 \text{ mm} \). Solving Eq. (5.1) for \( I \) and substituting the values of \( M \), \( \sigma_{all} \), and \( c \), we write

\[ I = \frac{M\overline{y}}{\sigma_{all}} = \frac{(1000 \text{ kN} \cdot \text{m})(355 \text{ mm})}{160 \text{ MPa}} = 2.219 \times 10^{-3} \text{ m}^4 = 2219 \times 10^6 \text{ mm}^4 \]

Replacing \( I \) by this value in Eq. (2) and solving for \( b \), we have

\[ 2219 \times 10^6 \text{ mm}^4 = 1190 \times 10^6 \text{ mm}^4 + (3.854 \times 10^6 \text{ mm}^4)b \]

\[ b = 267 \text{ mm} \]
### PROBLEMS

#### 5.126 and 5.127

The beam $AB$, consisting of an aluminum plate of uniform thickness $b$ and length $L$, is to support the load shown. 

(a) Knowing that the beam is to be of constant strength, express $h$ in terms of $x$, $L$, and $h_0$ for portion $AC$ of the beam. 

(b) Determine the maximum allowable load if $L = 800$ mm, $h_0 = 200$ mm, $b = 25$ mm, and $\sigma_{\text{all}} = 72$ MPa.

![Fig. P5.126](image)

![Fig. P5.127](image)

#### 5.128 and 5.129

The beam $AB$, consisting of a cast-iron plate of uniform thickness $b$ and length $L$, is to support the load shown. 

(a) Knowing that the beam is to be of constant strength, express $h$ in terms of $x$, $L$, and $h_0$. 

(b) Determine the maximum allowable load if $L = 36$ in., $h_0 = 12$ in., $b = 1.25$ in., and $\sigma_{\text{all}} = 24$ ksi.

![Fig. P5.128](image)

![Fig. P5.129](image)

#### 5.130 and 5.131

The beam $AB$, consisting of a cast-iron plate of uniform thickness $b$ and length $L$, is to support the distributed load $w(x)$ shown. 

(a) Knowing that the beam is to be of constant strength, express $h$ in terms of $x$, $L$, and $h_0$. 

(b) Determine the smallest value of $h_0$ if $L = 750$ mm, $b = 30$ mm, $w_0 = 300$ kN/m, and $\sigma_{\text{all}} = 200$ MPa.

![Fig. P5.130](image)

![Fig. P5.131](image)
A preliminary design on the use of a simply supported prismatic timber beam indicated that a beam with a rectangular cross section 50 mm wide and 200 mm deep would be required to safely support the load shown in part (a) of the figure. It was then decided to replace that beam with a built-up beam obtained by gluing together, as shown in part (b) of the figure, four pieces of the same timber as the original beam and of 50 × 50-mm cross section. Determine the length $l$ of the two outer pieces of timber that will yield the same factor of safety as the original design.

![Fig. P5.132](image1)

![Fig. P5.133](image2)

A preliminary design on the use of a cantilever prismatic timber beam indicated that a beam with a rectangular cross section 2 in. wide and 10 in. deep would be required to safely support the load shown in part (a) of the figure. It was then decided to replace that beam with a built-up beam obtained by gluing together, as shown in part (b) of the figure, five pieces of the same timber as the original beam and of 2 × 2-in. cross section. Determine the respective lengths $l_1$ and $l_2$ of the two inner and outer pieces of timber that will yield the same factor of safety as the original design.

![Fig. P5.134](image3)

![Fig. P5.135](image4)
5.136 and 5.137 A machine element of cast aluminum and in the shape of a solid of revolution of variable diameter \(d\) is being designed to support the load shown. Knowing that the machine element is to be of constant strength, express \(d\) in terms of \(x, L,\) and \(d_0\).

![Diagram](Fig. P5.136)

Fig. P5.136

![Diagram](Fig. P5.137)

Fig. P5.137

5.138 A cantilever beam \(AB\) consisting of a steel plate of uniform depth \(h\) and variable width \(b\) is to support the distributed load \(w\) along its centerline \(AB\). (a) Knowing that the beam is to be of constant strength, express \(b\) in terms of \(x, L,\) and \(b_0\). (b) Determine the maximum allowable value of \(w\) if \(L = 15\) in., \(b_0 = 8\) in., \(h = 0.75\) in., \(s = 24\) ksi.

![Diagram](Fig. P5.138)

Fig. P5.138

5.139 A cantilever beam \(AB\) consisting of a steel plate of uniform depth \(h\) and variable width \(b\) is to support the concentrated load \(P\) at point \(A\). (a) Knowing that the beam is to be of constant strength, express \(b\) in terms of \(x, L,\) and \(b_0\). (b) Determine the smallest allowable value of \(h\) if \(L = 300\) mm, \(b_0 = 375\) mm, \(P = 14.4\) kN, and \(s_{all} = 160\) MPa.

![Diagram](Fig. P5.139)

Fig. P5.139

5.140 Assuming that the length and width of the cover plates used with the beam of Sample Prob. 5.12 are, respectively, \(l = 4\) m and \(b = 285\) mm, and recalling that the thickness of each plate is \(16\) mm, determine the maximum normal stress on a transverse section (a) through the center of the beam, (b) just to the left of \(D\).
5.141 Knowing that $\sigma_{\text{all}} = 150 \text{ MPa}$, determine the largest concentrated load $P$ that can be applied at end $E$ of the beam shown.

![Beam Diagram](image)

5.142 Two cover plates, each $\frac{5}{8}$ in. thick, are welded to a W30 $\times$ 99 beam as shown. Knowing that $l = 9$ ft and $b = 12$ in., determine the maximum normal stress on a transverse section (a) through the center of the beam, (b) just to the left of $D$.

![Beam Diagram](image)

5.143 Two cover plates, each $\frac{5}{8}$ in. thick, are welded to a W30 $\times$ 99 beam as shown. Knowing that $\sigma_{\text{all}} = 22$ ksi for both the beam and the plates, determine the required value of (a) the length of the plates, (b) the width of the plates.

5.144 Two cover plates, each 7.5 mm thick, are welded to a W460 $\times$ 74 beam as shown. Knowing that $l = 5$ m and $b = 200$ mm, determine the maximum normal stress on a transverse section (a) through the center of the beam, (b) just to the left of $D$.

![Beam Diagram](image)

5.145 Two cover plates, each 7.5 mm thick, are welded to a W460 $\times$ 74 beam as shown. Knowing that $\sigma_{\text{all}} = 150$ MPa for both the beam and the plates, determine the required value of (a) the length of the plates, (b) the width of the plates.
5.146 Two cover plates, each \( \frac{1}{2} \) in. thick, are welded to a W27 \( \times \) 84 beam as shown. Knowing that \( l = 10 \text{ ft} \) and \( b = 10.5 \text{ in.} \), determine the maximum normal stress on a transverse section (a) through the center of the beam, (b) just to the left of D.

![Diagram of beam with plates](image)

Fig. P5.146 and P5.147

5.147 Two cover plates, each \( \frac{1}{2} \) in. thick, are welded to a W27 \( \times \) 84 beam as shown. Knowing that \( \sigma_{\text{all}} = 24 \text{ ksi} \) for both the beam and the plates, determine the required value of (a) the length of the plates, (b) the width of the plates.

5.148 For the tapered beam shown, determine (a) the transverse section in which the maximum normal stress occurs, (b) the largest distributed load \( w \) that can be applied, knowing that \( \sigma_{\text{all}} = 140 \text{ MPa} \).

5.149 For the tapered beam shown, knowing that \( w = 160 \text{ kN/m} \), determine (a) the transverse section in which the maximum normal stress occurs, (b) the corresponding value of the normal stress.

5.150 For the tapered beam shown, determine (a) the transverse section in which the maximum normal stress occurs, (b) the largest distributed load \( w \) that can be applied, knowing that \( \sigma_{\text{all}} = 24 \text{ ksi} \).

5.151 For the tapered beam shown, determine (a) the transverse section in which the maximum normal stress occurs, (b) the largest concentrated load \( P \) that can be applied, knowing that \( \sigma_{\text{all}} = 24 \text{ ksi} \).
This chapter was devoted to the analysis and design of beams under transverse loadings. Such loadings can consist of concentrated loads or distributed loads and the beams themselves are classified according to the way they are supported (Fig. 5.22). Only \textit{statically determinate} beams were considered in this chapter, where all support reactions can be determined by statics. The analysis of statically indeterminate beams is postponed until Chap. 9.

Considerations for the design of prismatic beams

While transverse loadings cause both bending and shear in a beam, the normal stresses caused by bending are the dominant criterion in the design of a beam for strength [Sec. 5.1]. Therefore, this chapter dealt only with the determination of the normal stresses in a beam, the effect of shearing stresses being examined in the next one.

We recalled from Sec. 4.4 the flexure formula for the determination of the maximum value $\sigma_m$ of the normal stress in a given section of the beam,

$$\sigma_m = \frac{|M|c}{I}$$  \hspace{1cm} (5.1)

where $I$ is the moment of inertia of the cross section with respect to a centroidal axis perpendicular to the plane of the bending couple $M$ and $c$ is the maximum distance from the neutral surface (Fig. 5.23). We also recalled from Sec. 4.4 that, introducing the elastic section modulus $S = I/c$ of the beam, the maximum value $\sigma_m$ of the normal stress in the section can be expressed as

$$\sigma_m = \frac{|M|}{S}$$  \hspace{1cm} (5.3)

It follows from Eq. (5.1) that the maximum normal stress occurs in the section where $|M|$ is largest, at the point farthest from the neutral

### Normal stresses due to bending

![Diagram](image)

**Fig. 5.23**

**Shear and bending-moment diagrams**

![Diagram](image)

**Fig. 5.24**
axis. The determination of the maximum value of $|M|$ and of the critical section of the beam in which it occurs is greatly simplified if we draw a shear diagram and a bending-moment diagram. These diagrams represent, respectively, the variation of the shear and of the bending moment along the beam and were obtained by determining the values of $V$ and $M$ at selected points of the beam [Sec. 5.2]. These values were found by passing a section through the point where they were to be determined and drawing the free-body diagram of either of the portions of beam obtained in this fashion. To avoid any confusion regarding the sense of the shearing force $V$ and of the bending couple $M$ (which act in opposite sense on the two portions of the beam), we followed the sign convention adopted earlier in the text as illustrated in Fig. 5.24 [Examples 5.01 and 5.02, Sample Probs. 5.1 and 5.2].

The construction of the shear and bending-moment diagrams is facilitated if the following relations are taken into account [Sec. 5.3]. Denoting by $w$ the distributed load per unit length (assumed positive if directed downward), we wrote

$$\frac{dV}{dx} = -w \quad \frac{dM}{dx} = V \quad (5.5, 5.7)$$

or, in integrated form,

$$V_D - V_C = -(\text{area under load curve between } C \text{ and } D) \quad (5.6')$$

$$M_D - M_C = \text{area under shear curve between } C \text{ and } D \quad (5.8')$$

Equation (5.6') makes it possible to draw the shear diagram of a beam from the curve representing the distributed load on that beam and the value of $V$ at one end of the beam. Similarly, Eq. (5.8') makes it possible to draw the bending-moment diagram from the shear diagram and the value of $M$ at one end of the beam. However, concentrated loads introduce discontinuities in the shear diagram and concentrated couples in the bending-moment diagram, none of which is accounted for in these equations [Sample Probs. 5.3 and 5.6]. Finally, we noted from Eq. (5.7) that the points of the beam where the bending moment is maximum or minimum are also the points where the shear is zero [Sample Prob. 5.4].

A proper procedure for the design of a prismatic beam was described in Sec. 5.4 and is summarized here:

Having determined $\sigma_{all}$ for the material used and assuming that the design of the beam is controlled by the maximum normal stress in the beam, compute the minimum allowable value of the section modulus:

$$S_{min} = \frac{|M|_{max}}{\sigma_{all}} \quad (5.9)$$

For a timber beam of rectangular cross section, $S = \frac{1}{6}bh^2$, where $b$ is the width of the beam and $h$ its depth. The dimensions of the section, therefore, must be selected so that $\frac{1}{6}bh^2 \geq S_{min}$.

For a rolled-steel beam, consult the appropriate table in Appendix C. Of the available beam sections, consider only those with a
section modulus \( S \geq S_{\text{min}} \) and select from this group the section with the smallest weight per unit length. This is the most economical of the sections for which \( S \geq S_{\text{min}} \).

**Singularity functions**

In Sec. 5.5, we discussed an alternative method for the determination of the maximum values of the shear and bending moment based on the use of the singularity functions \( \langle x - a \rangle^n \). By definition, and for \( n \geq 0 \), we had

\[
\langle x - a \rangle^n = \begin{cases} 
(x - a)^n & \text{when } x \geq a \\
0 & \text{when } x < a 
\end{cases}
\]  \( (5.14) \)

**Step function**

We noted that whenever the quantity between brackets is positive or zero, the brackets should be replaced by ordinary parentheses, and whenever that quantity is negative, the bracket itself is equal to zero. We also noted that singularity functions can be integrated and differentiated as ordinary binomials. Finally, we observed that the singularity function corresponding to \( n = 0 \) is discontinuous at \( x = a \) (Fig. 5.25). This function is called the step function. We wrote

\[
\langle x - a \rangle^0 = \begin{cases} 
1 & \text{when } x \geq a \\
0 & \text{when } x < a 
\end{cases}
\]  \( (5.15) \)

The use of singularity functions makes it possible to represent the shear or the bending moment in a beam by a single expression, valid at any point of the beam. For example, the contribution to the shear of the concentrated load \( P \) applied at the midpoint \( C \) of a simply supported beam (Fig. 5.26) can be represented by \(-P \langle x - \frac{1}{2}L \rangle^0\), since
this expression is equal to zero to the left of $C$, and to $-P$ to the right of $C$. Adding the contribution of the reaction $R_A = \frac{1}{2}P$ at $A$, we express the shear at any point of the beam as

$$V(x) = \frac{1}{2}P - P(x - \frac{1}{2}L)^0$$

The bending moment is obtained by integrating this expression:

$$M(x) = \frac{1}{2}Px - P(x - \frac{1}{2}L)^1$$

The singularity functions representing, respectively, the load, shear, and bending moment corresponding to various basic loadings were given in Fig. 5.18 on page 353. We noted that a distributed loading that does not extend to the right end of the beam, or which is discontinuous, should be replaced by an equivalent combination of open-ended loadings. For instance, a uniformly distributed load extending from $x = a$ to $x = b$ (Fig. 5.27) should be expressed as

$$w(x) = w_0(x - a)^0 - w_0(x - b)^0$$

The contribution of this load to the shear and to the bending moment can be obtained through two successive integrations. Care should be taken, however, to also include in the expression for $V(x)$ the contribution of concentrated loads and reactions, and to include in the expression for $M(x)$ the contribution of concentrated couples [Examples 5.05 and 5.06, Sample Probs. 5.9 and 5.10]. We also observed that singularity functions are particularly well suited to the use of computers.

We were concerned so far only with prismatic beams, i.e., beams of uniform cross section. Considering in Sec. 5.6 the design of nonprismatic beams, i.e., beams of variable cross section, we saw that by selecting the shape and size of the cross section so that its elastic section modulus $S = I/c$ varied along the beam in the same way as the bending moment $M$, we were able to design beams for which $\sigma_m$ at each section was equal to $\sigma_{all}$. Such beams, called beams of constant strength, clearly provide a more effective use of the material than prismatic beams. Their section modulus at any section along the beam was defined by the relation

$$S = \frac{M}{\sigma_{all}} \quad (5.18)$$
5.152 Draw the shear and bending-moment diagrams for the beam and loading shown, and determine the maximum absolute value (a) of the shear, (b) of the bending moment.

![Fig. P5.152](image)

5.153 Draw the shear and bending-moment diagrams for the beam and loading shown and determine the maximum normal stress due to bending.

![Fig. P5.153](image)

5.154 Determine (a) the distance \(a\) for which the maximum absolute value of the bending moment in the beam is as small as possible, (b) the corresponding maximum normal stress due to bending. (See hint of Prob. 5.27.)

![Fig. P5.154](image)
5.155 Determine (a) the equations of the shear and bending-moment curves for the beam and loading shown, (b) the maximum absolute value of the bending moment in the beam.

\[ w = w_0 \left( 1 + \frac{x^2}{L^2} \right) \]

![Figure P5.155](http://www.opoosoft.com)

5.156 Draw the shear and bending-moment diagrams for the beam and loading shown and determine the maximum normal stress due to bending.

![Figure P5.156](http://www.opoosoft.com)

5.157 Knowing that beam AB is in equilibrium under the loading shown, draw the shear and bending-moment diagrams and determine the maximum normal stress due to bending.

![Figure P5.157](http://www.opoosoft.com)
5.158 For the beam and loading shown, design the cross section of the beam, knowing that the grade of timber used has an allowable normal stress of 1750 psi.

![Fig. P5.158]

5.159 Knowing that the allowable stress for the steel used is 160 MPa, select the most economical wide-flange beam to support the loading shown.

![Fig. P5.159]

5.160 Determine the largest permissible value of $P$ for the beam and loading shown, knowing that the allowable normal stress is +8 ksi in tension and −18 ksi in compression.

![Fig. P5.160]
5.161 (a) Using singularity functions, find the magnitude and location of the maximum bending moment for the beam and loading shown. (b) Determine the maximum normal stress due to bending.

\[ \text{Fig. P5.161} \]

5.162 The beam AB, consisting of an aluminum plate of uniform thickness \( b \) and length \( L \), is to support the load shown. (a) Knowing that the beam is to be of constant strength, express \( h \) in terms of \( x \), \( L \), and \( h_0 \) for portion AC of the beam. (b) Determine the maximum allowable load if \( L = 800 \text{ mm} \), \( h_0 = 200 \text{ mm} \), \( b = 25 \text{ mm} \), and \( \sigma_{\text{all}} = 72 \text{ MPa} \).

\[ \text{Fig. P5.162} \]

5.163 A transverse force \( P \) is applied as shown at end A of the conical taper AB. Denoting by \( d_0 \) the diameter of the taper at A, show that the maximum normal stress occurs at point H, which is contained in a transverse section of diameter \( d = 1.5 d_0 \).

\[ \text{Fig. P5.163} \]
COMPUTER PROBLEMS

The following problems are designed to be solved with a computer.

5.C1 Several concentrated loads \( P_i \) \( (i = 1, 2, \ldots, n) \) can be applied to a beam as shown. Write a computer program that can be used to calculate the shear, bending moment, and normal stress at any point of the beam for a given loading of the beam and a given value of its section modulus. Use this program to solve Probs. 5.18, 5.21, and 5.25. (Hint: Maximum values will occur at a support or under a load.)

5.C2 A timber beam is to be designed to support a distributed load and up to two concentrated loads as shown. One of the dimensions of its uniform rectangular cross section has been specified and the other is to be determined so that the maximum normal stress in the beam will not exceed a given allowable value \( \sigma_{all} \). Write a computer program that can be used to calculate at given intervals \( \Delta L \) the shear, the bending moment, and the smallest acceptable value of the unknown dimension. Apply this program to solve the following problems, using the intervals \( \Delta L \) indicated: (a) Prob. 5.65 \( (\Delta L = 0.1 \text{ m}) \), (b) Prob. 5.69 \( (\Delta L = 0.3 \text{ m}) \), (c) Prob. 5.70 \( (\Delta L = 0.2 \text{ m}) \).

5.C3 Two cover plates, each of thickness \( t \), are to be welded to a wide-flange beam of length \( L \) that is to support a uniformly distributed load \( w \). Denoting by \( \sigma_{all} \) the allowable normal stress in the beam and in the plates, by \( d \) the depth of the beam, and by \( I_b \) and \( S_b \), respectively, the moment of inertia and the section modulus of the cross section of the unreinforced beam about a horizontal centroidal axis, write a computer program that can be used to calculate the required value of (a) the length \( a \) of the plates, (b) the width \( b \) of the plates. Use this program to solve Prob. 5.145.
5.C4 Two 25-kip loads are maintained 6 ft apart as they are moved slowly across the 18-ft beam AB. Write a computer program and use it to calculate the bending moment under each load and at the midpoint C of the beam for values of x from 0 to 24 ft at intervals $\Delta x = 1.5$ ft.

![Fig. P5.C4](image)

5.C5 Write a computer program that can be used to plot the shear and bending-moment diagrams for the beam and loading shown. Apply this program with a plotting interval $\Delta L = 0.2$ ft to the beam and loading of (a) Prob. 5.72, (b) Prob. 5.115.

![Fig. P5.C5](image)

5.C6 Write a computer program that can be used to plot the shear and bending-moment diagrams for the beam and loading shown. Apply this program with a plotting interval $\Delta L = 0.025$ m to the beam and loading of Prob. 5.112.

![Fig. P5.C6](image)
A reinforced concrete deck will be attached to each of the steel sections shown to form a composite box girder bridge. In this chapter the shearing stresses will be determined in various types of beams and girders.
CHAPTER 6

Shearing Stresses in Beams and Thin-Walled Members
Chapter 6 Shearing Stresses in Beams and Thin-Walled Members

6.1 Introduction

6.2 Shear on the Horizontal Face of a Beam Element

6.3 Determination of the Shearing Stresses in a Beam

6.4 Shearing Stresses $\tau_{xy}$ in Common Types of Beams

*6.5 Further Discussion of the Distribution of Stresses in a Narrow Rectangular Beam

6.6 Longitudinal Shear on a Beam Element of Arbitrary Shape

6.7 Shearing Stresses in Thin-Walled Members

*6.8 Plastic Deformations

*6.9 Unsymmetric Loading of Thin-Walled Members; Shear Center

6.1 INTRODUCTION

You saw in Sec. 5.1 that a transverse loading applied to a beam will result in normal and shearing stresses in any given transverse section of the beam. The normal stresses are created by the bending couple $M$ in that section and the shearing stresses by the shear $V$. Since the dominant criterion in the design of a beam for strength is the maximum value of the normal stress in the beam, our analysis was limited in Chap. 5 to the determination of the normal stresses. Shearing stresses, however, can be important, particularly in the design of short, stubby beams, and their analysis will be the subject of the first part of this chapter.

Figure 6.1 expresses graphically that the elementary normal and shearing forces exerted on a given transverse section of a prismatic beam with a vertical plane of symmetry are equivalent to the bending couple $M$ and the shearing force $V$. Six equations can be written to express that fact. Three of these equations involve only the normal forces $\sigma_x dA$ and have already been discussed in Sec. 4.2; they are Eqs. (4.1), (4.2), and (4.3), which express that the sum of the normal forces is zero and that the sums of their moments about the $y$ and $z$ axes are equal to zero and $M$, respectively. Three more equations involving the shearing forces $\tau_{xy} dA$ and $\tau_{xz} dA$ can now be written. One of them expresses that the sum of the moments of the shearing forces about the $x$ axis is zero and can be dismissed as trivial in view of the symmetry of the beam with respect to the $xy$ plane. The other two involve the $y$ and $z$ components of the elementary forces and are

$$y \text{ components: } \int \tau_{xy} dA = -V \quad (6.1)$$

$$z \text{ components: } \int \tau_{xz} dA = 0 \quad (6.2)$$

The first of these equations shows that vertical shearing stresses must exist in a transverse section of a beam under transverse loading. The second equation indicates that the average horizontal shearing stress in any section is zero. However, this does not mean that the shearing stress $\tau_{xz}$ is zero everywhere.

Let us now consider a small cubic element located in the vertical plane of symmetry of the beam (where we know that $\tau_{xz}$ must be zero) and examine the stresses exerted on its faces (Fig. 6.2). As we
have just seen, a normal stress $\sigma_x$ and a shearing stress $\tau_{xy}$ are exerted on each of the two faces perpendicular to the $x$ axis. But we know from Chap. 1 that, when shearing stresses $\tau_{xy}$ are exerted on the vertical faces of an element, equal stresses must be exerted on the horizontal faces of the same element. We thus conclude that longitudinal shearing stresses must exist in any member subjected to a transverse loading. This can be verified by considering a cantilever beam made of separate planks clamped together at one end (Fig. 6.3a). When a transverse load $P$ is applied to the free end of this composite beam, the planks are observed to slide with respect to each other (Fig. 6.3b). In contrast, if a couple $M$ is applied to the free end of the same composite beam (Fig. 6.3c), the various planks will bend into concentric arcs of circle and will not slide with respect to each other, thus verifying the fact that shear does not occur in a beam subjected to pure bending (cf. Sec. 4.3).

While sliding does not actually take place when a transverse load $P$ is applied to a beam made of a homogeneous and cohesive material such as steel, the tendency to slide does exist, showing that stresses occur on horizontal longitudinal planes as well as on vertical transverse planes. In the case of timber beams, whose resistance to shear is weaker between fibers, failure due to shear will occur along a longitudinal plane rather than a transverse plane (Photo 6.1).

In Sec. 6.2, a beam element of length $\Delta x$ bounded by two transverse planes and a horizontal one will be considered and the shearing force $\Delta H$ exerted on its horizontal face will be determined, as well as the shear per unit length, $q$, also known as $shear \ flow$. A formula for the shearing stress in a beam with a vertical plane of symmetry will be derived in Sec. 6.3 and used in Sec. 6.4 to determine the shearing stresses in common types of beams. The distribution of stresses in a narrow rectangular beam will be further discussed in Sec. 6.5.

The derivation given in Sec. 6.2 will be extended in Sec. 6.6 to cover the case of a beam element bounded by two transverse planes and a curved surface. This will allow us in Sec. 6.7 to determine the shearing stresses at any point of a symmetric thin-walled member, such as the flanges of wide-flange beams and box beams. The effect of plastic deformations on the magnitude and distribution of shearing stresses will be discussed in Sec. 6.8.
In the last section of the chapter (Sec. 6.9), the unsymmetric loading of thin-walled members will be considered and the concept of shear center will be introduced. You will then learn to determine the distribution of shearing stresses in such members.

### 6.2 SHEAR ON THE HORIZONTAL FACE OF A BEAM ELEMENT

Consider a prismatic beam $AB$ with a vertical plane of symmetry that supports various concentrated and distributed loads (Fig. 6.4). At a distance $x$ from end $A$ we detach from the beam an element $CDD'$ of length $\Delta x$ extending across the width of the beam from the upper surface of the beam to a horizontal plane located at a distance $y_1$ from the neutral axis (Fig. 6.5). The forces exerted on this element consist of vertical shearing forces $V_C$ and $V_D$, a horizontal shearing force $\Delta H$ exerted on the lower face of the element, elementary horizontal normal forces $\sigma_C dA$ and $\sigma_D dA$, and possibly a load $w \Delta x$ (Fig. 6.6). We write the equilibrium equation

$$\overrightarrow{\Sigma F_x} = 0: \quad \Delta H + \int_{\alpha} (\sigma_C - \sigma_D) dA = 0$$

where the integral extends over the shaded area $\alpha$ of the section located above the line $y = y_1$. Solving this equation for $\Delta H$ and using Eq. (5.2) of Sec. 5.1, $\sigma = My/I$, to express the normal stresses in terms of the bending moments at $C$ and $D$, we have

$$\Delta H = \frac{M_D - M_C}{I} \int_{\alpha} y \, dA \quad (6.3)$$
The integral in (6.3) represents the first moment with respect to the neutral axis of the portion $\alpha$ of the cross section of the beam that is located above the line $y = y_1$ and will be denoted by $Q$. On the other hand, recalling Eq. (5.7) of Sec. 5.3, we can express the increment $M_D - M_C$ of the bending moment as

$$M_D - M_C = \Delta M = (dM/dx) \Delta x = V \Delta x$$

Substituting into (6.3), we obtain the following expression for the horizontal shear exerted on the beam element

$$\Delta H = \frac{VQ}{I} \Delta x \quad (6.4)$$

The same result would have been obtained if we had used as a free body the lower element $C'D'D'C''$, rather than the upper element $CDD'C'$ (Fig. 6.7), since the shearing forces $\Delta H$ and $\Delta H'$ exerted by the two elements on each other are equal and opposite. This leads us to observe that the first moment $Q$ of the portion $\alpha'$ of the cross section located below the line $y = y_1$ (Fig. 6.7) is equal in magnitude and opposite in sign to the first moment of the portion $\alpha$ located above that line (Fig. 6.5). Indeed, the sum of these two moments is equal to the moment of the area of the entire cross section with respect to its centroidal axis and, thus, must be zero. This property can sometimes be used to simplify the computation of $Q$. We also note that $Q$ is maximum for $y_1 = 0$, since the elements of the cross section located above the neutral axis contribute positively to the integral in (6.3) that defines $Q$, while the elements located below that axis contribute negatively.

The horizontal shear per unit length, which will be denoted by the letter $q$, is obtained by dividing both members of Eq. (6.4) by $\Delta x$:

$$q = \frac{\Delta H}{\Delta x} = \frac{VQ}{I} \quad (6.5)$$

We recall that $Q$ is the first moment with respect to the neutral axis of the portion of the cross section located either above or below the point at which $q$ is being computed, and that $I$ is the centroidal moment of inertia of the entire cross-sectional area. For a reason that will become apparent later (Sec. 6.7), the horizontal shear per unit length $q$ is also referred to as the shear flow.
A beam is made of three planks, 20 by 100 mm in cross section, nailed together (Fig. 6.8). Knowing that the spacing between nails is 25 mm and that the vertical shear in the beam is \( V = 500 \text{ N} \), determine the shearing force in each nail.

We first determine the horizontal force per unit length, \( q \), exerted on the lower face of the upper plank. We use Eq. (6.5), where \( Q \) represents the first moment with respect to the neutral axis of the shaded area \( A \) shown in Fig. 6.9a, and where \( I \) is the moment of inertia about the same axis of the entire cross-sectional area (Fig. 6.9b). Recalling that the first moment of an area with respect to a given axis is equal to the product of the area and of the distance from its centroid to the axis, we have

\[
Q = A \bar{y} = (0.020 \text{ m} \times 0.100 \text{ m})(0.060 \text{ m}) = 120 \times 10^{-6} \text{ m}^3
\]

\[
I = \frac{1}{6}(0.020 \text{ m})(0.100 \text{ m})^3 + 2\left[\frac{1}{12}(0.100 \text{ m})(0.020 \text{ m})^3 + (0.020 \text{ m} \times 0.100 \text{ m})(0.060 \text{ m})^2\right]
= 1.667 \times 10^{-6} + 2(0.0667 + 7.2)10^{-6}
= 16.20 \times 10^{-6} \text{ m}^4
\]

Substituting into Eq. (6.5), we write

\[
q = \frac{VQ}{I} = \frac{(500 \text{ N})(120 \times 10^{-6} \text{ m}^3)}{16.20 \times 10^{-6} \text{ m}^4} = 3704 \text{ N/m}
\]

Since the spacing between the nails is 25 mm, the shearing force in each nail is

\[
F = (0.025 \text{ m})q = (0.025 \text{ m})(3704 \text{ N/m}) = 92.6 \text{ N}
\]

### 6.3 Determination of the Shearing Stresses in a Beam

Consider again a beam with a vertical plane of symmetry, subjected to various concentrated or distributed loads applied in that plane. We saw in the preceding section that if, through two vertical cuts and one horizontal cut, we detach from the beam an element of length \( \Delta x \) (Fig. 6.10), the magnitude \( \Delta H \) of the shearing force exerted on the horizontal face of the element can be obtained from Eq. (6.4). The average shearing stress \( \tau_{\text{ave}} \) on that face of the element is obtained by dividing \( \Delta H \) by the area \( \Delta A \) of the face. Observing that \( \Delta A = t \Delta x \), where \( t \) is the width of the element at the cut, we write

\[
\tau_{\text{ave}} = \frac{\Delta H}{\Delta A} = \frac{VQ}{I} \frac{\Delta x}{t \Delta x}
\]

or

\[
\tau_{\text{ave}} = \frac{VQ}{It}
\]

\(^1\text{See Appendix A.}\)
We note that, since the shearing stresses $\tau_{xy}$ and $\tau_{yz}$ exerted respectively on a transverse and a horizontal plane through $D'$ are equal, the expression obtained also represents the average value of $\tau_{xy}$ along the line $D'D''$ (Fig. 6.11).

We observe that $\tau_{yz} = 0$ on the upper and lower faces of the beam, since no forces are exerted on these faces. It follows that $\tau_{xy} = 0$ along the upper and lower edges of the transverse section (Fig. 6.12). We also note that, while $Q$ is maximum for $y = 0$ (see Sec. 6.2), we cannot conclude that $\tau_{ave}$ will be maximum along the neutral axis, since $\tau_{ave}$ depends upon the width $t$ of the section as well as upon $Q$.

As long as the width of the beam cross section remains small compared to its depth, the shearing stress varies only slightly along the line $D'D''$ (Fig. 6.11) and Eq. (6.6) can be used to compute $\tau_{xy}$ at any point along $D'D''$. Actually, $\tau_{xy}$ is larger at points $D'_1$ and $D''_2$ than at $D'$, but the theory of elasticity shows† that, for a beam of rectangular section of width $b$ and depth $h$, and as long as $b \leq h/4$, the value of the shearing stress at points $C_1$ and $C_2$ (Fig. 6.13) does not exceed by more than 0.8% the average value of the stress computed along the neutral axis.‡

### 6.4 Shearing Stresses $\tau_{xy}$ in Common Types of Beams

We saw in the preceding section that, for a narrow rectangular beam, i.e., for a beam of rectangular section of width $b$ and depth $h$ with $b \leq h$, the variation of the shearing stress $\tau_{xy}$ across the width of the beam is less than 0.8% of $\tau_{ave}$. We can, therefore, use Eq. (6.6) in practical applications to determine the shearing stress at any point of the cross section of a narrow rectangular beam and write

$$\tau_{xy} = \frac{VQ}{It}$$

where $t$ is equal to the width $b$ of the beam, and where $Q$ is the first moment with respect to the neutral axis of the shaded area $A$ (Fig. 6.14).

Observing that the distance from the neutral axis to the centroid $C'$ of $A$ is $\bar{y} = \frac{1}{2}(c + y)$, and recalling that $Q = A\bar{y}$, we write

$$Q = A\bar{y} = bh(c - y)\frac{1}{2}(c + y) = \frac{1}{2}bh(c^2 - y^2)$$


‡On the other hand, for large values of $b/h$, the value $\tau_{max}$ of the stress at $C_1$ and $C_2$ may be many times larger than the average value $\tau_{ave}$ computed along the neutral axis, as we may see from the following table:

<table>
<thead>
<tr>
<th>$b/h$</th>
<th>0.25</th>
<th>0.5</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>10</th>
<th>20</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_{max}/\tau_{ave}$</td>
<td>1.008</td>
<td>1.033</td>
<td>1.126</td>
<td>1.396</td>
<td>1.988</td>
<td>2.582</td>
<td>3.770</td>
<td>6.740</td>
<td>15.65</td>
</tr>
<tr>
<td>$\tau_{ave}/\tau_{ave}$</td>
<td>0.996</td>
<td>0.983</td>
<td>0.940</td>
<td>0.856</td>
<td>0.805</td>
<td>0.800</td>
<td>0.800</td>
<td>0.800</td>
<td>0.800</td>
</tr>
</tbody>
</table>
Recalling, on the other hand, that \( I = \frac{bh^3}{12} = \frac{3}{4}bc^3 \), we have

\[
\tau_{xy} = \frac{VQ}{Ib} = \frac{3}{4} \frac{c^3 - y^2}{bc^3} V
\]

or, noting that the cross-sectional area of the beam is \( A = 2bc \),

\[
\tau_{xy} = \frac{3}{2} \frac{V}{A} \left( 1 - \frac{y^2}{c^2} \right) \quad (6.9)
\]

Equation (6.9) shows that the distribution of shearing stresses in a transverse section of a rectangular beam is parabolic (Fig. 6.15). As we have already observed in the preceding section, the shearing stresses are zero at the top and bottom of the cross section \( (y = \pm c) \). Making \( y = 0 \) in Eq. (6.9), we obtain the value of the maximum shearing stress in a given section of a narrow rectangular beam:

\[
\tau_{\text{max}} = \frac{3}{2} \frac{V}{A} \quad (6.10)
\]

The relation obtained shows that the maximum value of the shearing stress in a beam of rectangular cross section is 50% larger than the value \( V/A \) that would be obtained by wrongly assuming a uniform stress distribution across the entire cross section.

In the case of an American standard beam (S-beam) or a wide-flange beam (W-beam), Eq. (6.6) can be used to determine the average value of the shearing stress \( \tau_{\text{xy}} \) over a section \( aa' \) or \( bb' \) of the transverse cross section of the beam (Figs. 6.16a and b). We write

\[
\tau_{\text{ave}} = \frac{VQ}{It} \quad (6.6)
\]

where \( V \) is the vertical shear, \( t \) the width of the section at the elevation considered, \( Q \) the first moment of the shaded area with respect to the neutral axis \( cc' \), and \( I \) the moment of inertia of the entire cross-sectional area about \( cc' \). Plotting \( \tau_{\text{ave}} \) against the vertical distance \( y \), we obtain the curve shown in Fig. 6.16c. We note the discontinuities existing in this curve, which reflect the difference between the values of \( t \) corresponding respectively to the flanges \( ABDG \) and \( A'B'C'D' \) and to the web \( EFF'E' \).

In the case of the web, the shearing stress \( \tau_{xy} \) varies only very slightly across the section \( bb' \), and can be assumed equal to its average

\[
\tau_{\text{ave}}
\]
value \( \tau_{xy} \). This is not true, however, for the flanges. For example, considering the horizontal line \( DEFG \), we note that \( \tau_{xy} \) is zero between \( D \) and \( E \) and between \( F \) and \( G \), since these two segments are part of the free surface of the beam. On the other hand the value of \( \tau_{xy} \) between \( E \) and \( F \) can be obtained by making \( t = EF \) in Eq. (6.6). In practice, one usually assumes that the entire shear load is carried by the web, and that a good approximation of the maximum value of the shearing stress in the cross section can be obtained by dividing \( V \) by the cross-sectional area of the web.

\[
\tau_{\text{max}} = \frac{V}{A_{\text{web}}} \quad (6.11)
\]

We should note, however, that while the vertical component \( \tau_{xy} \) of the shearing stress in the flanges can be neglected, its horizontal component \( \tau_{xz} \) has a significant value that will be determined in Sec. 6.7.

EXAMPLE 6.02

Knowing that the allowable shearing stress for the timber beam of Sample Prob. 5.7 is \( \tau_{\text{all}} = 0.250 \text{ ksi} \), check that the design obtained in that sample problem is acceptable from the point of view of the shearing stresses.

We recall from the shear diagram of Sample Prob. 5.7 that \( V_{\text{max}} = 4.50 \text{ kips} \). The actual width of the beam was given as \( b = 3.5 \text{ in.} \), and the value obtained for its depth was \( h = 14.55 \text{ in.} \). Using Eq. (6.10) for the maximum shearing stress in a narrow rectangular beam, we write

\[
\tau_{\text{max}} = \frac{3V}{2A} = \frac{3 \times 4.50 \text{ kips}}{2 \times (3.5 \text{ in.})(14.55 \text{ in.})} = 0.1325 \text{ ksi}
\]

Since \( \tau_{\text{max}} < \tau_{\text{all}} \), the design obtained in Sample Prob. 5.7 is acceptable.

EXAMPLE 6.03

Knowing that the allowable shearing stress for the steel beam of Sample Prob. 5.8 is \( \tau_{\text{all}} = 90 \text{ MPa} \), check that the W360 \( \times \) 32.9 shape obtained in that sample problem is acceptable from the point of view of the shearing stresses.

We recall from the shear diagram of Sample Prob. 5.8 that the maximum absolute value of the shear in the beam is \( |V|_{\text{max}} = 58 \text{ kN} \). As we saw in Sec. 6.4, it may be assumed in practice that the entire shear load is carried by the web and that the maximum value of the shearing stress in the beam can be obtained from Eq. (6.11). From Appendix C we find that for a W360 \( \times \) 32.9 shape the depth of the beam and the thickness of its web are, respectively, \( d = 349 \text{ mm} \) and \( t_w = 5.8 \text{ mm} \). We thus have

\[
A_{\text{web}} = d t_w = (349 \text{ mm})(5.8 \text{ mm}) = 2024 \text{ mm}^2
\]

Substituting the values of \( |V|_{\text{max}} \) and \( A_{\text{web}} \) into Eq. (6.11), we obtain

\[
\tau_{\text{max}} = \frac{|V|_{\text{max}}}{A_{\text{web}}} = \frac{58 \text{ kN}}{2024 \text{ mm}^2} = 28.7 \text{ MPa}
\]

Since \( \tau_{\text{max}} < \tau_{\text{all}} \), the design obtained in Sample Prob. 5.8 is acceptable.
FURTHER DISCUSSION OF THE DISTRIBUTION OF STRESSES IN A NARROW RECTANGULAR BEAM

Consider a narrow cantilever beam of rectangular cross section of width \( b \) and depth \( h \) subjected to a load \( P \) at its free end (Fig. 6.17). Since the shear \( V \) in the beam is constant and equal in magnitude to the load \( P \), Eq. (6.9) yields

\[
\tau_{xy} = \frac{3}{2} \frac{P}{A} \left( 1 - \frac{y^2}{c^2} \right)
\]  

(6.12)

Fig. 6.18  Deformation of segment of cantilever beam.

We note from Eq. (6.12) that the shearing stresses depend only upon the distance \( y \) from the neutral surface. They are independent, therefore, of the distance from the point of application of the load; it follows that all elements located at the same distance from the neutral surface undergo the same shear deformation (Fig. 6.18). While plane sections do not remain plane, the distance between two corresponding points \( D \) and \( D' \) located in different sections remains the same. This indicates that the normal strains \( \epsilon_n \), and thus the normal stresses \( \sigma_n \), are unaffected by the shearing stresses, and that the assumption made in Sec. 5.1 is justified for the loading condition of Fig. 6.17.

We conclude that our analysis of the stresses in a cantilever beam of rectangular cross section, subjected to a concentrated load \( P \) at its free end, is valid. The correct values of the shearing stresses in the beam are given by Eq. (6.12), and the normal stresses at a distance \( x \) from the free end are obtained by making \( M = -Px \) in Eq. (5.2) of Sec. 5.1. We have

\[
\sigma_n = + \frac{Pxy}{I}
\]  

(6.13)

The validity of the above statement, however, depends upon the end conditions. If Eq. (6.12) is to apply everywhere, then the load \( P \) must be distributed parabolically over the free-end section. Moreover, the fixed-end support must be of such a nature that it will allow the type of shear deformation indicated in Fig. 6.18.
resulting model (Fig. 6.19) is highly unlikely to be encountered in practice. However, it follows from Saint-Venant’s principle that, for other modes of application of the load and for other types of fixed-end supports, Eqs. (6.12) and (6.13) still provide us with the correct distribution of stresses, except close to either end of the beam.

When a beam of rectangular cross section is subjected to several concentrated loads (Fig. 6.20), the principle of superposition can be used to determine the normal and shearing stresses in sections located between the points of application of the loads. However, since the loads $P_2, P_3, \ldots$, are applied on the surface of the beam and cannot be assumed to be distributed parabolically throughout the cross section, the results obtained cease to be valid in the immediate vicinity of the points of application of the loads.

When the beam is subjected to a distributed load (Fig. 6.21), the shear varies with the distance from the end of the beam, and so does the shearing stress at a given elevation $y$. The resulting shear deformations are such that the distance between two corresponding points of different cross sections, such as $D_1$ and $D_1'$, or $D_2$ and $D_2'$, will depend upon their elevation. This indicates that the assumption that plane sections remain plane, under which Eqs. (6.12) and (6.13) were derived, must be rejected for the loading condition of Fig. 6.21. The error involved, however, is small for the values of the span-depth ratio encountered in practice.

We should also note that, in portions of the beam located under a concentrated or distributed load, normal stresses $\sigma_y$ will be exerted on the horizontal faces of a cubic element of material, in addition to the stresses $\tau_{xy}$ shown in Fig. 6.2.
SAMPLE PROBLEM 6.1

Beam AB is made of three planks glued together and is subjected, in its plane of symmetry, to the loading shown. Knowing that the width of each glued joint is 20 mm, determine the average shearing stress in each joint at section n-n of the beam. The location of the centroid of the section is given in the sketch and the centroidal moment of inertia is known to be \( I = 8.63 \times 10^{-6} \text{ m}^4 \).

SOLUTION

Vertical Shear at Section n-n. Since the beam and loading are both symmetric with respect to the center of the beam, we have \( A = B = 1.5 \text{ kN} \uparrow \).

Shearing Stress in Joint a. We pass the section \( a-a \) through the glued joint and separate the cross-sectional area into two parts. We choose to determine \( Q \) by computing the first moment with respect to the neutral axis of the area above section \( a-a \).

\[
Q = A\bar{y}_1 = [(0.100 \text{ m})(0.020 \text{ m})](0.0417 \text{ m}) = 83.4 \times 10^{-6} \text{ m}^3
\]

Recalling that the width of the glued joint is \( t = 0.020 \text{ m} \), we use Eq. (6.7) to determine the average shearing stress in the joint.

\[
\tau_{ave} = \frac{VQ}{It} = \frac{(1500 \text{ N})(83.4 \times 10^{-6} \text{ m}^3)}{(8.63 \times 10^{-6} \text{ m}^3)(0.020 \text{ m})} \tau_{ave} = 725 \text{ kPa}
\]

Shearing Stress in Joint b. We now pass section \( b-b \) and compute \( Q \) by using the area below the section.

\[
Q = A\bar{y}_2 = [(0.060 \text{ m})(0.020 \text{ m})](0.0583 \text{ m}) = 70.0 \times 10^{-6} \text{ m}^3
\]

\[
\tau_{ave} = \frac{VQ}{It} = \frac{(1500 \text{ N})(70.0 \times 10^{-6} \text{ m}^3)}{(8.63 \times 10^{-6} \text{ m}^3)(0.020 \text{ m})} \tau_{ave} = 608 \text{ kPa}
\]
SAMPLE PROBLEM 6.2

A timber beam AB of span 10 ft and nominal width 4 in. (actual width = 3.5 in.) is to support the three concentrated loads shown. Knowing that for the grade of timber used $\sigma_{all} = 1800$ psi and $\tau_{all} = 120$ psi, determine the minimum required depth $d$ of the beam.

**SOLUTION**

Maximum Shear and Bending Moment. After drawing the shear and bending-moment diagrams, we note that

$$M_{\text{max}} = 7.5 \text{ kip} \cdot \text{ft} = 90 \text{ kip} \cdot \text{in.}$$

$$V_{\text{max}} = 3 \text{ kips}$$

Design Based on Allowable Normal Stress. We first express the elastic section modulus $S$ in terms of the depth $d$. We have

$$I = \frac{1}{12} bd^3$$

$$S = \frac{1}{c} = \frac{1}{6} = \frac{1}{6} (3.5)d^2 = 0.5833d^2$$

For $M_{\text{max}} = 90 \text{ kip} \cdot \text{in.}$ and $\sigma_{\text{all}} = 1800$ psi, we write

$$S = \frac{M_{\text{max}}}{\sigma_{\text{all}}} = \frac{90 \times 10^3 \text{ lb} \cdot \text{in.}}{1800 \text{ psi}} = 85.7$$

$$d^2 = \frac{85.7}{d} = 9.26 \text{ in.}$$

We have satisfied the requirement that $\sigma_{\text{all}} \leq 1800$ psi.

Check Shearing Stress. For $V_{\text{max}} = 3 \text{ kips}$ and $d = 9.26 \text{ in.}$, we find

$$\tau_{\text{m}} = \frac{3}{2} \frac{V_{\text{max}}}{A} = \frac{3}{2} \frac{3000 \text{ lb}}{(3.5 \text{ in.})(9.26 \text{ in.})} = 138.8 \text{ psi}$$

Since $\tau_{\text{all}} = 120$ psi, the depth $d = 9.26$ in. is *not* acceptable and we must redesign the beam on the basis of the requirement that $\tau_{\text{m}} \leq 120$ psi.

Design Based on Allowable Shearing Stress. Since we now know that the allowable shearing stress controls the design, we write

$$\tau_{\text{m}} = \tau_{\text{all}} = \frac{3}{2} \frac{V_{\text{max}}}{A} = 120 \text{ psi} = \frac{3}{2} \frac{3000 \text{ lb}}{(3.5 \text{ in.})d}$$

$$d = 10.71 \text{ in.}$$

The normal stress is, of course, less than $\sigma_{\text{all}} = 1800$ psi, and the depth of 10.71 in. is fully acceptable.

Comment. Since timber is normally available in depth increments of 2 in., a 4 $\times$ 12-in. nominal size timber should be used. The actual cross section would then be 3.5 $\times$ 11.25 in.
6.1 Three boards, each of 1.5 × 3.5-in. rectangular cross section, are nailed together to form a beam that is subjected to a vertical shear of 250 lb. Knowing that the spacing between each pair of nails is 2.5 in., determine the shearing force in each nail.

Fig. P6.1

6.2 Three boards, each 2 in. thick, are nailed together to form a beam that is subjected to a vertical shear. Knowing that the allowable shearing force in each nail is 150 lb, determine the allowable shear if the spacing $s$ between the nails is 3 in.

Fig. P6.2

6.3 Three boards are nailed together to form a beam shown, which is subjected to a vertical shear. Knowing that the spacing between the nails is $s = 75$ mm and that the allowable shearing force in each nail is 400 N, determine the allowable shear when $w = 120$ mm.

6.4 Solve Prob. 6.3, assuming that the width of the top and bottom boards is changed to $w = 100$ mm.
6.5 The American Standard rolled-steel beam shown has been reinforced by attaching to it two \( 16 \times 200 \text{-mm} \) plates, using 18-mm-diameter bolts spaced longitudinally every 120 mm. Knowing that the average allowable shearing stress in the bolts is 90 MPa, determine the largest permissible vertical shearing force.

![Fig. P6.5](image)

6.6 Solve Prob. 6.5, assuming that the reinforcing plates are only 12 mm thick.

6.7 A column is fabricated by connecting the rolled-steel members shown by bolts of \( \frac{3}{8} \text{-in.} \) diameter spaced longitudinally every 5 in. Determine the average shearing stress in the bolts caused by a shearing force of 30 kips parallel to the \( y \) axis.

![Fig. P6.7](image)

6.8 The composite beam shown is fabricated by connecting two \( W6 \times 20 \) rolled-steel members, using bolts of \( \frac{3}{8} \text{-in.} \) diameter spaced longitudinally every 6 in. Knowing that the average allowable shearing stress in the bolts is 10.5 ksi, determine the largest allowable vertical shear in the beam.

![Fig. P6.8](image)
6.9 through 6.12 For the beam and loading shown, consider section $n-n$ and determine (a) the largest shearing stress in that section, (b) the shearing stress at point $a$.

**Fig. P6.9**

Dimensions in mm

**Fig. P6.11**

**Fig. P6.12**
6.13 and 6.14 For a beam having the cross section shown, determine the largest allowable vertical shear if the shearing stress is not to exceed 60 MPa.

![Fig. P6.13](image1)

6.15 For the beam and loading shown, determine the minimum required depth \( h \), knowing that for the grade of timber used, \( \sigma_{\text{all}} = 1750 \text{ psi} \) and \( \tau_{\text{all}} = 130 \text{ psi} \).

![Fig. P6.15](image2)

6.16 For the beam and loading shown, determine the minimum required width \( b \), knowing that for the grade of timber used, \( \sigma_{\text{all}} = 12 \text{ MPa} \) and \( \tau_{\text{all}} = 825 \text{ kPa} \).

![Fig. P6.16](image3)

6.17 A timber beam \( AB \) of length \( L \) and rectangular cross section carries a uniformly distributed load \( w \) and is supported as shown. (a) Show that the ratio \( \tau_m/\sigma_m \) of the maximum values of the shearing and normal stresses in the beam is equal to \( 2h/L \), where \( h \) and \( L \) are, respectively, the depth and the length of the beam. (b) Determine the depth \( h \) and the width \( b \) of the beam, knowing that \( L = 5 \text{ m} \), \( w = 8 \text{ kN/m} \), \( \tau_m = 1.08 \text{ MPa} \), and \( \sigma_m = 12 \text{ MPa} \).

![Fig. P6.17](image4)
6.18 A timber beam \( AB \) of length \( L \) and rectangular cross section carries a single concentrated load \( P \) at its midpoint \( C \). (a) Show that the ratio \( \tau_m/\sigma_m \) of the maximum values of the shearing and normal stresses in the beam is equal to \( h/2L \), where \( h \) and \( L \) are, respectively, the depth and the length of the beam. (b) Determine the depth \( h \) and the width \( b \) of the beam, knowing that \( L = 2 \text{ m}, P = 40 \text{ kN}, \tau_m = 960 \text{ kPa}, \) and \( \sigma_m = 12 \text{ MPa} \).

6.19 For the wide-flange beam with the loading shown, determine the largest \( P \) that can be applied, knowing that the maximum normal stress is 24 ksi and the largest shearing stress using the approximation \( \tau_m = V/A_{web} \) is 14.5 ksi.

6.20 For the wide-flange beam with the loading shown, determine the largest load \( P \) that can be applied, knowing that the maximum normal stress is 160 MPa and the largest shearing stress using the approximation \( \tau_m = V/A_{web} \) is 100 MPa.

6.21 and 6.22 For the beam and loading shown, consider section \( m-n \) and determine the shearing stress at (a) point \( a \), (b) point \( b \).

6.23 and 6.24 For the beam and loading shown, determine the largest shearing stress in section \( n-n \).
**6.25 through 6.28** A beam having the cross section shown is subjected to a vertical shear $V$. Determine $(a)$ the horizontal line along which the shearing stress is maximum, $(b)$ the constant $k$ in the following expression for the maximum shearing stress

$$\tau_{\text{max}} = k \frac{V}{A}$$

where $A$ is the cross-sectional area of the beam.

![Fig. P6.25](image)

![Fig. P6.26](image)

![Fig. P6.27](image)

![Fig. P6.28](image)

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**6.6 LONGITUDINAL SHEAR ON A BEAM ELEMENT OF ARBITRARY SHAPE**

Consider a box beam obtained by nailing together four planks, as shown in Fig. 6.22a. You learned in Sec. 6.2 how to determine the shear per unit length, $q$, on the horizontal surfaces along which the planks are joined. But could you determine $q$ if the planks had been joined along vertical surfaces, as shown in Fig. 6.22b? We examined in Sec. 6.4 the distribution of the vertical components $t_{yz}$ of the stresses on a transverse section of a W-beam or an S-beam and found that these stresses had a fairly constant value in the web of the beam and were negligible in its flanges. But what about the horizontal components $t_{xz}$ of the stresses in the flanges?

To answer these questions we must extend the procedure developed in Sec. 6.2 for the determination of the shear per unit length, $q$, so that it will apply to the cases just described.

![Fig. 6.22 Box beam cross sections.](image)

Consider the prismatic beam $AB$ of Fig. 6.4, which has a vertical plane of symmetry and supports the loads shown. At a distance $x$ from end $A$ we detach again an element $CDD'C'$ of length $\Delta x$. This element, however, will now extend from two sides of the beam.
Shearing Stresses in Beams and Thin-Walled Members

to an arbitrary curved surface (Fig. 6.23). The forces exerted on the element include vertical shearing forces \( V_C \) and \( V_D \), elementary horizontal normal forces \( \sigma_C \, dA \) and \( \sigma_D \, dA \), possibly a load \( w \, \Delta x \), and a longitudinal shearing force \( \Delta H \) representing the resultant of the elementary longitudinal shearing forces exerted on the curved surface (Fig. 6.24). We write the equilibrium equation

\[
\sum F_x = 0: \quad \Delta H + \int \limits_\alpha (\sigma_C - \sigma_D) \, dA = 0
\]

where the integral is to be computed over the shaded area \( \alpha \) of the section. We observe that the equation obtained is the same as the one we obtained in Sec. 6.2, but that the shaded area \( \alpha \) over which the integral is to be computed now extends to the curved surface.

The remainder of the derivation is the same as in Sec. 6.2. We find that the longitudinal shear exerted on the beam element is

\[
\Delta H = \frac{VQ}{I} \Delta x \quad (6.4)
\]

where \( I \) is the centroidal moment of inertia of the entire section, \( Q \) the first moment of the shaded area \( \alpha \) with respect to the neutral axis, and \( V \) the vertical shear in the section. Dividing both members of Eq. (6.4) by \( \Delta x \), we obtain the horizontal shear per unit length, or shear flow:

\[
q = \frac{\Delta H}{\Delta x} = \frac{VQ}{I} \quad (6.5)
\]
6.7 SHEARING STRESSES IN THIN-WALLED MEMBERS

We saw in the preceding section that Eq. (6.4) may be used to determine the longitudinal shear $\Delta H$ exerted on the walls of a beam element of arbitrary shape and Eq. (6.5) to determine the corresponding shear flow $q$. These equations will be used in this section to calculate both the shear flow and the average shearing stress in thin-walled
members such as the flanges of wide-flange beams (Photo 6.2) and box beams, or the walls of structural tubes (Photo 6.3).

Consider, for instance, a segment of length $\Delta x$ of a wide-flange beam (Fig. 6.27a) and let $V$ be the vertical shear in the transverse section shown. Let us detach an element $ABB'A'$ of the upper flange (Fig. 6.27b). The longitudinal shear $\Delta H$ exerted on that element can be obtained from Eq. (6.4):

$$\Delta H = \frac{VQ}{I} \Delta x$$  \hspace{1cm} (6.4)

Dividing $\Delta H$ by the area $\Delta A = t \Delta x$ of the cut, we obtain for the average shearing stress exerted on the element the same expression that we had obtained in Sec. 6.3 in the case of a horizontal cut:

$$\tau_{ave} = \frac{VQ}{It}$$  \hspace{1cm} (6.6)

Note that $\tau_{ave}$ now represents the average value of the shearing stress $\tau_{zx}$ over a vertical cut, but since the thickness $t$ of the flange is small, there is very little variation of $\tau_{zx}$ across the cut. Recalling that $\tau_{zx} = \tau_{zx}$ (Fig. 6.28), we conclude that the horizontal component $\tau_{zx}$ of the
shearing stress at any point of a transverse section of the flange can be obtained from Eq. (6.6), where \( Q \) is the first moment of the shaded area about the neutral axis (Fig. 6.29a). We recall that a similar result was obtained in Sec. 6.4 for the vertical component \( \tau_{xy} \) of the shearing stress in the web (Fig. 6.29b). Equation (6.6) can be used to determine shearing stresses in box beams (Fig. 6.30), half pipes (Fig. 6.31), and other thin-walled members, as long as the loads are applied in a plane of symmetry of the member. In each case, the cut must be perpendicular to the surface of the member, and Eq. (6.6) will yield the component of the shearing stress in the direction of the tangent to that surface. (The other component may be assumed equal to zero, in view of the proximity of the two free surfaces.)

Comparing Eqs. (6.5) and (6.6), we note that the product of the shearing stress \( \tau \) at a given point of the section and of the thickness \( t \) of the section at that point is equal to \( q \). Since \( V \) and \( I \) are constant in any given section, \( q \) depends only upon the first moment \( Q \) and, thus, can easily be sketched on the section. In the case of a box beam, for example (Fig. 6.32), we note that \( q \) grows smoothly from zero at \( A \) to a maximum value at \( C \) and \( C' \) on the neutral axis, and then decreases back to zero as \( E \) is reached. We also note that there is no sudden variation in the magnitude of \( q \) as we pass a corner at \( B, D, B', \) or \( D' \), and that the sense of \( q \) in the horizontal portions of the section may be easily obtained from its sense in the vertical portions (which is the same as the sense of the shear \( V \)). In the case of a wide-flange section (Fig. 6.33), the values of \( q \) in portions \( AB \) and \( A'B' \) of the upper flange are distributed symmetrically. As we turn at \( B \) into the web, the values of \( q \) corresponding to the two halves of the flange must be combined to obtain the value of \( q \) at the top of the web. After reaching a maximum value at \( C \) on the neutral axis, \( q \) decreases, and at \( D \) splits into two equal parts corresponding to the two halves of the lower flange. The name of shear flow commonly used to refer to the shear per unit length, \( q \), reflects the similarity between the properties of \( q \) that we have just described and some of the characteristics of a fluid flow through an open channel or pipe.†

†We recall that the concept of shear flow was used to analyze the distribution of shearing stresses in thin-walled hollow shafts (Sec. 3.13). However, while the shear flow in a hollow shaft is constant, the shear flow in a member under a transverse loading is not.
So far we have assumed that all the loads were applied in a plane of symmetry of the member. In the case of members possessing two planes of symmetry, such as the wide-flange beam of Fig. 6.29 or the box beam of Fig. 6.30, any load applied through the centroid of a given cross section can be resolved into components along the two axes of symmetry of the section. Each component will cause the member to bend in a plane of symmetry, and the corresponding shearing stresses can be obtained from Eq. (6.6). The principle of superposition can then be used to determine the resulting stresses.

However, if the member considered possesses no plane of symmetry, or if it possesses a single plane of symmetry and is subjected to a load that is not contained in that plane, the member is observed to bend and twist at the same time, except when the load is applied at a specific point, called the shear center. Note that the shear center generally does not coincide with the centroid of the cross section. The determination of the shear center of various thin-walled shapes is discussed in Sec. 6.9.

*6.8 PLASTIC DEFORMATIONS*

Consider a cantilever beam AB of length L and rectangular cross section, subjected at its free end A to a concentrated load P (Fig. 6.34). The largest value of the bending moment occurs at the fixed end B and is equal to $M = PL$. As long as this value does not exceed the maximum elastic moment $M_y$, the normal stress $\sigma_x$ will not exceed the yield strength $\sigma_y$ anywhere in the beam. However, as $P$ is increased beyond the value $M_y/L$, yield is initiated at points B and B' and spreads toward the free end of the beam. Assuming the material to be elastoplastic, and considering a cross section CC' located at a distance $x$ from the free end A of the beam (Fig. 6.35), we obtain the half-thickness $y_Y$ of the elastic core in that section by making $M = Px$ in Eq. (4.38) of Sec. 4.9. We have

$$P_x = \frac{3}{2} M_y \left(1 - \frac{1}{3} \frac{y_Y^2}{c^2}\right)$$

(6.14)

where $c$ is the half-depth of the beam. Plotting $y_Y$ against $x$, we obtain the boundary between the elastic and plastic zones.
As long as \( PL < \frac{3}{2} M_Y \), the parabola defined by Eq. (6.14) intersects the line \( BB' \), as shown in Fig. 6.38. However, when \( PL \) reaches the value \( \frac{3}{2} M_Y \), that is, when \( PL = M_p \), where \( M_p \) is the plastic moment defined in Sec. 4.9, Eq. (6.14) yields \( y = 0 \) for \( x = L \), which shows that the vertex of the parabola is now located in section \( BB' \), and that this section has become fully plastic (Fig. 6.36). Recalling Eq. (4.40) of Sec. 4.9, we also note that the radius of curvature \( \rho \) of the neutral surface at that point is equal to zero, indicating the presence of a sharp bend in the beam at its fixed end. We say that a plastic hinge has developed at that point. The load \( P = M_p / L \) is the largest load that can be supported by the beam.

The above discussion was based only on the analysis of the normal stresses in the beam. Let us now examine the distribution of the shearing stresses in a section that has become partly plastic. Consider the portion of beam \( CC''D''D \) located between the transverse sections \( CC' \) and \( DD' \), and above the horizontal plane \( D''C'' \) (Fig. 6.37a). If this portion is located entirely in the plastic zone, the normal stresses exerted on the faces \( CC'' \) and \( DD'' \) will be uniformly distributed and equal to the yield strength \( \sigma_Y \) (Fig. 6.40b). The equilibrium of the free body \( CC''D''D \) thus requires that the horizontal shearing force \( \Delta H \) exerted on its lower face be equal to zero. It follows that the average value of the horizontal shearing stress \( \tau_x \) across the beam at \( C'' \) is zero, as well as the average value of the vertical shearing stress \( \tau_y \). We thus conclude that the vertical shear \( V = P \) in section \( CC'' \) must be distributed entirely over the portion \( EE' \) of that section that is located within the elastic zone (Fig. 6.38). It can be shown† that the distribution of the shearing stresses over \( EE' \) is the same as in an elastic rectangular beam of the same width \( b \) as beam \( AB \), and of depth equal to the thickness \( 2y \) of the elastic zone. Denoting by \( A' \) the area \( 2by \) of the elastic portion of the cross section, we have

\[
\tau_y = \frac{3}{2} \frac{P}{A'} \left( 1 - \frac{y_y^2}{y} \right)
\]  

(6.15)

The maximum value of the shearing stress occurs for \( y = 0 \) and is

\[
\tau_{\text{max}} = \frac{3}{2} \frac{P}{A'}
\]  

(6.16)

As the area \( A' \) of the elastic portion of the section decreases, \( \tau_{\text{max}} \) increases and eventually reaches the yield strength in shear \( \tau_Y \). Thus, shear contributes to the ultimate failure of the beam. A more exact analysis of this mode of failure should take into account the combined effect of the normal and shearing stresses.

†See Prob. 6.60.
SAMPLE PROBLEM 6.3

Knowing that the vertical shear is 50 kips in a W10 × 68 rolled-steel beam, determine the horizontal shearing stress in the top flange at a point a located 4.31 in. from the edge of the beam. The dimensions and other geometric data of the rolled-steel section are given in Appendix C.

SOLUTION

We isolate the shaded portion of the flange by cutting along the dashed line that passes through point a.

\[ Q = (4.31 \text{ in.})(0.770 \text{ in.})(4.815 \text{ in.}) = 15.98 \text{ in}^3 \]

\[ \tau = \frac{VQ}{It} = \frac{(50 \text{ kips})(15.98 \text{ in}^3)}{(394 \text{ in}^4)(0.770 \text{ in.})} \]

\[ \tau = 2.63 \text{ ksi} \]

SAMPLE PROBLEM 6.4

Solve Sample Prob. 6.3, assuming that 0.75 × 12-in. plates have been attached to the flanges of the W10 × 68 beam by continuous fillet welds as shown.

SOLUTION

For the composite beam the centroidal moment of inertia is

\[ I = 394 \text{ in}^4 + 2[\frac{1}{12}(12 \text{ in.})(0.75 \text{ in.})^3 + (12 \text{ in.})(0.75 \text{ in.})(5.575 \text{ in.})^2] \]

\[ I = 954 \text{ in}^4 \]

Since the top plate and the flange are connected only at the welds, we find the shearing stress at a by passing a section through the flange at a, between the plate and the flange, and again through the flange at the symmetric point a'.

For the shaded area that we have isolated, we have

\[ t = 2t_f = 2(0.770 \text{ in.}) = 1.540 \text{ in.} \]

\[ Q = 2[(4.31 \text{ in.})(0.770 \text{ in.})(4.815 \text{ in.})] + (12 \text{ in.})(0.75 \text{ in.})(5.575 \text{ in.}) \]

\[ Q = 82.1 \text{ in}^3 \]

\[ \tau = \frac{VQ}{It} = \frac{(50 \text{ kips})(82.1 \text{ in}^3)}{(954 \text{ in}^4)(1.540 \text{ in.})} \]

\[ \tau = 2.79 \text{ ksi} \]
SAMPLE PROBLEM 6.5

The thin-walled extruded beam shown is made of aluminum and has a uniform 3-mm wall thickness. Knowing that the shear in the beam is 5 kN, determine (a) the shearing stress at point A, (b) the maximum shearing stress in the beam.

Note: The dimensions given are to lines midway between the outer and inner surfaces of the beam.

SOLUTION

Centroid. We note that \( AB = AD = 65 \text{ mm} \).

\[
\bar{y} = \frac{\sum y A}{\sum A} = \frac{2[(65 \text{ mm})(3 \text{ mm})(30 \text{ mm})]}{2[(65 \text{ mm})(3 \text{ mm})] + (50 \text{ mm})(3 \text{ mm})} = 21.67 \text{ mm}
\]

Centroidal Moment of Inertia. Each side of the thin-walled beam can be considered as a parallelogram, and we recall that for the case shown

\[
I = \bar{y}^2 + \frac{bh^3}{12}
\]

where \( b \) is measured parallel to the axis \( nn \).

\[
b = (3 \text{ mm})/\cos \beta = (3 \text{ mm})/(12/13) = 3.25 \text{ mm}
\]

\[
I = \Sigma(I + Ad^2) = 2\left[\frac{1}{12}(3.25 \text{ mm})(60 \text{ mm})^3 + (3.25 \text{ mm})(60 \text{ mm})(8.33 \text{ mm})^2 + \frac{1}{4}(50 \text{ mm})(3 \text{ mm})^3 + (50 \text{ mm})(3 \text{ mm})(21.67 \text{ mm})^2\right]
\]

\[
I = 214.6 \times 10^3 \text{ mm}^4 \quad I = 0.2146 \times 10^{-6} \text{ m}^4
\]

a. Shearing Stress at \( A \). If a shearing stress \( t_A \) occurs at \( A \), the shear flow will be \( q_A = \sigma t_A \) and must be directed in one of the two ways shown. But the cross section and the loading are symmetric about a vertical line through \( A \), and thus the shear flow must also be symmetric. Since neither of the possible shear flows is symmetric, we conclude that \( t_A = 0 \).

b. Maximum Shearing Stress. Since the wall thickness is constant, the maximum shearing stress occurs at the neutral axis, where \( Q \) is maximum. Since we know that the shearing stress at \( A \) is zero, we cut the section along the dashed line shown and isolate the shaded portion of the beam. In order to obtain the largest shearing stress, the cut at the neutral axis is made perpendicular to the sides, and is of length \( t = 3 \text{ mm} \).

\[
Q = \left[(3.25 \text{ mm})(38.33 \text{ mm})\right] \left(\frac{38.33 \text{ mm}}{2}\right) = 2387 \text{ mm}^3
\]

\[
Q = 2.387 \times 10^{-6} \text{ m}^3
\]

\[
\tau_E = \frac{VQ}{It} = \frac{(5 \text{ kN})(2.387 \times 10^{-6} \text{ m}^3)}{(0.2146 \times 10^{-6} \text{ m}^3)(0.003 \text{ m})} = 18.54 \text{ MPa}
\]
6.29 The built-up beam shown is made by gluing together five planks. Knowing that in the glued joints the average allowable shearing stress is 350 kPa, determine the largest permissible vertical shear in the beam.

6.30 For the beam of Prob. 6.29, determine the largest permissible horizontal shear.

6.31 Several wooden planks are glued together to form the box beam shown. Knowing that the beam is subjected to a vertical shear of 3 kN, determine the average shearing stress in the glued joint (a) at A, (b) at B.

6.32 The built-up timber beam is subjected to a 1500-lb vertical shear. Knowing that the longitudinal spacing of the nails is \( s = 2.5 \) in. and that each nail is 3.5 in. long, determine the shearing force in each nail.

6.33 The built-up wooden beam shown is subjected to a vertical shear of 8 kN. Knowing that the nails are spaced longitudinally every 60 mm at A and every 25 mm at B, determine the shearing force in the nails (a) at A, (b) at B. (Given: \( I_x = 1.504 \times 10^9 \) mm^4.)
6.34 Knowing that a vertical shear $V$ of 50 kips is exerted on a W14 × 82 rolled-steel beam, determine the shearing stress (a) at point $a$, (b) at the centroid $C$.

6.35 An extruded aluminum beam has the cross section shown. Knowing that the vertical shear in the beam is 150 kN, determine the shearing stress at (a) point $a$, (b) point $b$.

6.36 Knowing that a given vertical shear $V$ causes a maximum shearing stress of 75 MPa in the hat-shaped extrusion shown, determine the corresponding shearing stress at (a) point $a$, (b) point $b$.

6.37 Knowing that a given vertical shear $V$ causes a maximum shearing stress of 75 MPa in an extruded beam having a cross section shown, determine the shearing stress at the three points indicated.

6.38 An extruded beam has the cross section shown and a uniform wall thickness of 0.20 in. Knowing that a given vertical shear $V$ causes a maximum shearing stress $\tau = 9$ ksi, determine the shearing stress at the four points indicated.

6.39 Solve Prob. 6.38 assuming that the beam is subjected to a horizontal shear $V$. 

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6.40 Knowing that a given vertical shear $V$ causes a maximum shearing stress of 50 MPa in a thin-walled member having the cross section shown, determine the corresponding shearing stress at (a) point $a$, (b) point $b$, (c) point $c$.

6.41 and 6.42 The extruded aluminum beam has a uniform wall thickness of $\frac{1}{8}$ in. Knowing that the vertical shear in the beam is 2 kips, determine the corresponding shearing stress at each of the five points indicated.

6.43 Three 1 $\times$ 18-in. steel plates are bolted to four L6 $\times$ 6 $\times$ 1 angles to form a beam with the cross section shown. The bolts have a $\frac{7}{8}$-in. diameter and are spaced longitudinally every 5 in. Knowing that the allowable average shearing stress in the bolts is 12 ksi, determine the largest permissible vertical shear in the beam. (Given: $I_x = 6123 \text{ in}^4$.)

6.44 Three planks are connected as shown by bolts of 14-mm diameter spaced every 150 mm along the longitudinal axis of the beam. For a vertical shear of 10 kN, determine the average shearing stress in the bolts.
6.45 A beam consists of three planks connected as shown by steel bolts with a longitudinal spacing of 225 mm. Knowing that the shear in the beam is vertical and equal to 6 kN and that the allowable average shearing stress in each bolt is 60 MPa, determine the smallest permissible bolt diameter that can be used.

Fig. P6.45

6.46 A beam consists of five planks of 1.5 × 6-in. cross section connected by steel bolts with a longitudinal spacing of 9 in. Knowing that the shear in the beam is vertical and equal to 2000 lb and that the allowable average shearing stress in each bolt is 7500 psi, determine the smallest permissible bolt diameter that can be used.

Fig. P6.46

6.47 A plate of ⅛-in. thickness is corrugated as shown and then used as a beam. For a vertical shear of 1.2 kips, determine (a) the maximum shearing stress in the section, (b) the shearing stress at point B. Also sketch the shear flow in the cross section.

Fig. P6.47

6.48 A plate of 4-mm thickness is bent as shown and then used as a beam. For a vertical shear of 12 kN, determine (a) the shearing stress at point A, (b) the maximum shearing stress in the beam. Also sketch the shear flow in the cross section.

Fig. P6.48

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6.49 A plate of 2-mm thickness is bent as shown and then used as a beam. For a vertical shear of 5 kN, determine the shearing stress at the five points indicated and sketch the shear flow in the cross section.

6.50 A plate of thickness \( t \) is bent as shown and then used as a beam. For a vertical shear of 600 lb, determine (a) the thickness \( t \) for which the maximum shearing stress is 300 psi, (b) the corresponding shearing stress at point \( E \). Also sketch the shear flow in the cross section.

6.51 The design of a beam calls for connecting two vertical rectangular \( \frac{3}{8} \times 4\text{-in.} \) plates by welding them to two horizontal \( \frac{1}{2} \times 2\text{-in.} \) plates as shown. For a vertical shear \( V \), determine the dimension \( a \) for which the shear flow through the welded surfaces is maximum.

6.52 and 6.53 An extruded beam has a uniform wall thickness \( t \). Denoting by \( V \) the vertical shear and by \( A \) the cross-sectional area of the beam, express the maximum shearing stress as \( \tau_{\text{max}} = k(V/A) \) and determine the constant \( k \) for each of the two orientations shown.

6.54 (a) Determine the shearing stress at point \( P \) of a thin-walled pipe of the cross section shown caused by a vertical shear \( V \). (b) Show that the maximum shearing stress occurs for \( \theta = 90^\circ \) and is equal to \( 2V/A \), where \( A \) is the cross-sectional area of the pipe.
6.55 For a beam made of two or more materials with different moduli of elasticity, show that Eq. (6.6)

\[ \tau_{ave} = \frac{VQ}{It} \]

remains valid provided that both \( Q \) and \( I \) are computed by using the transformed section of the beam (see Sec. 4.6) and provided further that \( t \) is the actual width of the beam where \( \tau_{ave} \) is computed.

6.56 and 6.57 A steel bar and an aluminum bar are bonded together as shown to form a composite beam. Knowing that the vertical shear in the beam is 4 kips and that the modulus of elasticity is \( 29 \times 10^6 \) psi for the steel and \( 10.6 \times 10^6 \) psi for the aluminum, determine (a) the average stress at the bonded surface, (b) the maximum shearing stress in the beam. (Hint: Use the method indicated in Prob. 6.55.)

6.58 and 6.59 A composite beam is made by attaching the timber and steel portions shown with bolts of 12-mm diameter spaced longitudinally every 200 mm. The modulus of elasticity is 10 GPa for the wood and 200 GPa for the steel. For a vertical shear of 4 kN, determine (a) the average shearing stress in the bolts, (b) the shearing stress at the center of the cross section. (Hint: Use the method indicated in Prob. 6.55.)
Consider the cantilever beam $AB$ discussed in Sec. 6.8 and the portion $ACKJ$ of the beam that is located to the left of the transverse section $CC'$ and above the horizontal plane $JK$, where $K$ is a point at a distance $y < y_Y$ above the neutral axis (Fig. P6.60).

(a) Recalling that $\sigma_x = \sigma_Y$ between $C$ and $E$ and $\sigma_x = (\sigma_Y/y_Y)y$ between $E$ and $K$, show that the magnitude of the horizontal shearing force $H$ exerted on the lower face of the portion of beam $ACKJ$ is

$$H = \frac{1}{2} b \sigma_Y \left( 2x - y_Y - \frac{y^2}{y_Y} \right)$$

(b) Observing that the shearing stress at $K$ is

$$\tau_{xy} = \lim_{\Delta A \to 0} \frac{\Delta H}{\Delta A} = \lim_{\Delta x \to 0} \frac{1}{b} \frac{\Delta H}{\Delta x} = \frac{1}{b} \frac{\partial H}{\partial x}$$

and recalling that $y_Y$ is a function of $x$ defined by Eq. (6.14), derive Eq. (6.15).

### 6.9 UNSYMMETRIC LOADING OF THIN-WALLED MEMBERS; SHEAR CENTER

Our analysis of the effects of transverse loadings in Chap. 5 and in the preceding sections of this chapter was limited to members possessing a vertical plane of symmetry and to loads applied in that plane. The members were observed to bend in the plane of loading (Fig. 6.39) and, in any given cross section, the bending couple $M$ and the shear $V$ (Fig. 6.40) were found to result in normal and shearing stresses defined, respectively, by the formulas

$$\sigma_x = -\frac{My}{I} \quad (4.16)$$

and

$$\tau_{ave} = \frac{VQ}{It} \quad (6.6)$$

In this section, the effects of transverse loadings on thin-walled members that do not possess a vertical plane of symmetry will be
examined. Let us assume, for example, that the channel member of Fig. 6.39 has been rotated through $90^\circ$ and that the line of action of $P$ still passes through the centroid of the end section. The couple vector $M$ representing the bending moment in a given cross section is still directed along a principal axis of the section (Fig. 6.41), and the neutral axis will coincide with that axis (cf. Sec. 4.13). Equation (4.16), therefore, is applicable and can be used to compute the normal stresses in the section. However, Eq. (6.6) cannot be used to determine the shearing stresses in the section, since this equation was derived for a member possessing a vertical plane of symmetry (cf. Sec. 6.7). Actually, the member will be observed to \textit{bend and twist} under the applied load (Fig. 6.42), and the resulting distribution of shearing stresses will be quite different from that defined by Eq. (6.6).

The following question now arises: Is it possible to apply the vertical load $P$ in such a way that the channel member of Fig. 6.42 will \textit{bend without twisting} and, if so, where should the load $P$ be applied? If the member bends without twisting, then the shearing stress at any point of a given cross section can be obtained from Eq. (6.6), where $Q$ is the first moment of the shaded area with respect to the neutral axis (Fig. 6.43a), and the distribution of stresses will look as shown in Fig. 6.43b, with $\tau = 0$ at both $A$ and $E$. We note that the shearing force exerted on a small element of cross-sectional area $dA = t \, ds$ is $dF = \tau \, dA = \tau t \, ds$, or $dF = q \, ds$ (Fig. 6.44a).
where \( q \) is the shear flow \( q = \tau = VQ/I \) at the point considered. The resultant of the shearing forces exerted on the elements of the upper flange \( AB \) of the channel is found to be a horizontal force \( F \) (Fig. 6.44b) of magnitude

\[
F = \int_{A}^{B} q \, ds \quad (6.17)
\]

Because of the symmetry of the channel section about its neutral axis, the resultant of the shearing forces exerted on the lower flange \( DE \) is a force \( F' \) of the same magnitude as \( F \) but of opposite sense. We conclude that the resultant of the shearing forces exerted on the web \( BD \) must be equal to the vertical shear \( V \) in the section:

\[
V = \int_{B}^{D} q \, ds \quad (6.18)
\]

We now observe that the forces \( F \) and \( F' \) form a couple of moment \( Fh \), where \( h \) is the distance between the center lines of the flanges \( AB \) and \( DE \) (Fig. 6.45a). This couple can be eliminated if the vertical shear \( V \) is moved to the left through a distance \( e \) such that the moment of \( V \) about \( B \) is equal to \( Fh \) (Fig. 6.45b). We write

\[
Ve = Fh \quad (6.19)
\]

and conclude that, when the force \( P \) is applied at a distance \( e \) to the left of the center line of the web \( BD \), the member bends in a vertical plane without twisting (Fig. 6.46).

The point \( O \) where the line of action of \( P \) intersects the axis of symmetry of the end section is called the shear center of that section. We note that, in the case of an oblique load \( P \) (Fig. 6.47a), the member will also be free of any twist if the load \( P \) is applied at the shear center of the section. Indeed, the load \( P \) can then be resolved into two components \( P_x \) and \( P_y \) (Fig. 6.47b) corresponding respectively to the loading conditions of Figs. 6.39 and 6.46, neither of which causes the member to twist.
EXAMPLE 6.05

Determine the shear center O of a channel section of uniform thickness (Fig. 6.48), knowing that \( b = 4 \) in., \( h = 6 \) in., and \( t = 0.15 \) in.

Assuming that the member does not twist, we first determine the shear flow \( q \) in flange \( AB \) at a distance \( s \) from \( A \) (Fig. 6.49). Recalling Eq. (6.5) and observing that the first moment \( Q \) of the shaded area with respect to the neutral axis is \( Q = (st)(h/2) \), we write

\[
q = \frac{VQ}{I} = \frac{Vsth}{2I}
\]  

(6.20)

where \( V \) is the vertical shear and \( I \) the moment of inertia of the section with respect to the neutral axis.

Recalling Eq. (6.17), we determine the magnitude of the shearing force \( F \) exerted on flange \( AB \) by integrating the shear flow \( q \) from \( A \) to \( B \):

\[
F = \int_{0}^{b} q \, ds = \int_{0}^{b} \frac{Vsth}{2I} \, ds = \frac{Vth}{2I} \int_{0}^{b} s \, ds
\]

\[
F = \frac{Vthb^{2}}{4I}
\]

(6.21)

The distance \( e \) from the center line of the web \( BD \) to the shear center \( O \) can now be obtained from Eq. (6.19):

\[
e = \frac{Fh}{V} = \frac{Vthb^{2}}{4I} \frac{h}{V} = \frac{th^{2}b^{2}}{4I}
\]

(6.22)

The moment of inertia \( I \) of the channel section can be expressed as follows:

\[
I = I_{\text{web}} + 2I_{\text{flange}}
\]

\[
= \frac{1}{12}th^{3} + 2 \left[ \frac{1}{12}tb^{3} + bt \left( \frac{h}{2} \right)^{2} \right]
\]

Neglecting the term containing \( t^{3} \), which is very small, we have

\[
I = \frac{1}{12}th^{3} + \frac{1}{2}tbh^{2} = \frac{1}{2}th^{3}(6b + h)
\]

(6.23)

Substituting this expression into (6.22), we write

\[
e = \frac{3b^{2}}{6b + h} = \frac{b}{2 + \frac{h}{3b}}
\]

(6.24)

We note that the distance \( e \) does not depend upon \( t \) and can vary from 0 to \( b/2 \), depending upon the value of the ratio \( h/3b \). For the given channel section, we have

\[
\frac{h}{3b} = \frac{6 \text{ in.}}{3(4 \text{ in.})} = 0.5
\]

and

\[
e = \frac{4 \text{ in.}}{2 + 0.5} = 1.6 \text{ in.}
\]
For the channel section of Example 6.05 determine the distribution of the shearing stresses caused by a 2.5-kip vertical shear \( V \) applied at the shear center \( O \) (Fig. 6.50).

**Shearing stresses in flanges.** Since \( V \) is applied at the shear center, there is no torsion, and the stresses in flange \( AB \) are obtained from Eq. (6.20) of Example 6.05. We have

\[
\tau = \frac{q}{t} = \frac{VQ}{It} = \frac{Vh}{2It}
\]  

(6.25)

which shows that the stress distribution in flange \( AB \) is linear. Letting \( s = b \) and substituting for \( I \) from Eq. (6.23), we obtain the value of the shearing stress at \( B \):

\[
\tau_B = \frac{Vhb}{2(\pi b^2)(6b + h)} = \frac{6Vb}{th(6b + h)}
\]  

(6.26)

Letting \( V = 2.5 \) kips, and using the given dimensions, we have

\[
\tau_B = \frac{6(2.5 \text{ kips})(4 \text{ in.})}{(0.15 \text{ in.})(6 \text{ in.})(6 \times 4 \text{ in.} + 6 \text{ in.})} = 2.22 \text{ ksi}
\]

**Shearing stresses in web.** The distribution of the shearing stresses in the web \( BD \) is parabolic, as in the case of a W-beam, and the maximum stress occurs at the neutral axis. Computing the first moment of the upper half of the cross section with respect to the neutral axis (Fig. 6.51), we write

\[
Q = \frac{b}{2} h^2, \quad I = \frac{bh^3}{12}
\]  

(6.27)

Substituting for \( I \) and \( Q \) from (6.23) and (6.27), respectively, into the expression for the shearing stress, we have

\[
\tau_{\text{max}} = \frac{VQ}{It} = \frac{V(\frac{b}{2} h^2)(4b + h)}{\pi th^2(6b + h)t} = \frac{3V(4b + h)}{2th(6b + h)}
\]

or, with the given data,

\[
\tau_{\text{max}} = \frac{3(2.5 \text{ kips})(4 \times 4 \text{ in.} + 6 \text{ in.})}{2(0.15 \text{ in.})(6 \text{ in.})(6 \times 4 \text{ in.} + 6 \text{ in.})} = 3.06 \text{ ksi}
\]

**Distribution of stresses over the section.** The distribution of the shearing stresses over the entire channel section has been plotted in Fig. 6.52.
For the channel section of Example 6.05, and neglecting stress concentrations, determine the maximum shearing stress caused by a 2.5-kip vertical shear $V$ applied at the centroid $C$ of the section, which is located 1.143 in. to the right of the center line of the web $BD$ (Fig. 6.53).

**Equivalent force-couple system at shear center.** The shear center $O$ of the cross section was determined in Example 6.05 and found to be at a distance $e = 1.6$ in. to the left of the center line of the web $BD$. We replace the shear $V$ (Fig. 6.54a) by an equivalent force-couple system at the shear center $O$ (Fig. 6.54b). This system consists of a 2.5-kip force $V$ and of a torque $T$ of magnitude

$$ T = V(OC) = (2.5 \text{ kips})(1.6 \text{ in.} + 1.143 \text{ in.}) $$

$$ = 6.86 \text{ kip} \cdot \text{in.} $$

**Stresses due to bending.** The 2.5-kip force $V$ causes the member to bend, and the corresponding distribution of shearing stresses in the section (Fig. 6.54c) was determined in Example 6.06. We recall that the maximum value of the stress due to this force was found to be

$$ \tau_{\text{max}}^{\text{bending}} = 3.06 \text{ ksi} $$

**Stresses due to twisting.** The torque $T$ causes the member to twist, and the corresponding distribution of stresses is shown in Fig. 6.54d. We recall from Sec. 3.12 that the membrane analogy shows that, in a thin-walled member of uniform thickness, the stress caused by a torque $T$ is maximum along the edge of the section. Using Eqs. (3.45) and (3.43) with

$$ a = 4 \text{ in.} + 6 \text{ in.} + 4 \text{ in.} = 14 \text{ in.} $$

$$ b = t = 0.15 \text{ in.} $$

we have

$$ c_1 = \frac{2}{3}(1 - 0.63 \times \frac{b}{a}) = \frac{1}{3}(1 - 0.63 \times 0.0107) = 0.331 $$

$$ (\tau_{\text{max}})^{\text{twisting}} = \frac{T}{c_1 ab^2} = \frac{6.86 \text{ kip} \cdot \text{in.}}{(0.331)(14 \text{ in.})(0.15 \text{ in.})^2} = 65.8 \text{ ksi} $$

**Combined stresses.** The maximum stress due to the combined bending and twisting occurs at the neutral axis, on the inside surface of the web, and is

$$ \tau_{\text{max}} = 3.06 \text{ ksi} + 65.8 \text{ ksi} = 68.9 \text{ ksi} $$

---

**Fig. 6.54**

---

**EXAMPLE 6.07**

For the channel section of Example 6.05, and neglecting stress concentrations, determine the maximum shearing stress caused by a 2.5-kip vertical shear $V$ applied at the centroid $C$ of the section, which is located 1.143 in. to the right of the center line of the web $BD$ (Fig. 6.53).

**Equivalent force-couple system at shear center.** The shear center $O$ of the cross section was determined in Example 6.05 and found to be at a distance $e = 1.6$ in. to the left of the center line of the web $BD$. We replace the shear $V$ (Fig. 6.54a) by an equivalent force-couple system at the shear center $O$ (Fig. 6.54b). This system consists of a 2.5-kip force $V$ and of a torque $T$ of magnitude

$$ T = V(OC) = (2.5 \text{ kips})(1.6 \text{ in.} + 1.143 \text{ in.}) $$

$$ = 6.86 \text{ kip} \cdot \text{in.} $$

**Stresses due to bending.** The 2.5-kip force $V$ causes the member to bend, and the corresponding distribution of shearing stresses in the section (Fig. 6.54c) was determined in Example 6.06. We recall that the maximum value of the stress due to this force was found to be

$$ \tau_{\text{max}}^{\text{bending}} = 3.06 \text{ ksi} $$

**Stresses due to twisting.** The torque $T$ causes the member to twist, and the corresponding distribution of stresses is shown in Fig. 6.54d. We recall from Sec. 3.12 that the membrane analogy shows that, in a thin-walled member of uniform thickness, the stress caused by a torque $T$ is maximum along the edge of the section. Using Eqs. (3.45) and (3.43) with

$$ a = 4 \text{ in.} + 6 \text{ in.} + 4 \text{ in.} = 14 \text{ in.} $$

$$ b = t = 0.15 \text{ in.} $$

we have

$$ c_1 = \frac{2}{3}(1 - 0.63 \times \frac{b}{a}) = \frac{1}{3}(1 - 0.63 \times 0.0107) = 0.331 $$

$$ (\tau_{\text{max}})^{\text{twisting}} = \frac{T}{c_1 ab^2} = \frac{6.86 \text{ kip} \cdot \text{in.}}{(0.331)(14 \text{ in.})(0.15 \text{ in.})^2} = 65.8 \text{ ksi} $$

**Combined stresses.** The maximum stress due to the combined bending and twisting occurs at the neutral axis, on the inside surface of the web, and is

$$ \tau_{\text{max}} = 3.06 \text{ ksi} + 65.8 \text{ ksi} = 68.9 \text{ ksi} $$

---

**Fig. 6.54**
Turning our attention to thin-walled members possessing no plane of symmetry, we now consider the case of an angle shape subjected to a vertical load $P$. If the member is oriented in such a way that the load $P$ is perpendicular to one of the principal centroidal axes $Cz$ of the cross section, the couple vector $M$ representing the bending moment in a given section will be directed along $Cz$ (Fig. 6.55), and the neutral axis will coincide with that axis (cf. Sec. 4.13). Equation (4.16), therefore, is applicable and can be used to compute the normal stresses in the section. We now propose to determine where the load $P$ should be applied if Eq. (6.6) is to define the shearing stresses in the section, i.e., if the member is to bend without twisting.

Let us assume that the shearing stresses in the section are defined by Eq. (6.6). As in the case of the channel member considered earlier, the elementary shearing forces exerted on the section can be expressed as $dF = q\, ds$, with $q = VQ/I$, where $Q$ represents a first moment with respect to the neutral axis (Fig. 6.56a). We note that the resultant of the shearing forces exerted on portion OA of the cross section is a force $F_1$ directed along OA, and that the resultant of the shearing forces exerted on portion OB is a force $F_2$ along OB (Fig. 6.56b). Since both $F_1$ and $F_2$ pass through point O at the corner of the angle, it follows that their own resultant, which is the shear $V$ in the section, must also pass through O (Fig. 6.56c). We conclude that the member will not be twisted if the line of action of the load $P$ passes through the corner $O$ of the section in which it is applied.

The same reasoning can be applied when the load $P$ is perpendicular to the other principal centroidal axis $Cy$ of the angle section. And, since any load $P$ applied at the corner $O$ of a cross section can be resolved into components perpendicular to the principal axes, it follows that the member will not be twisted if each load is applied at the corner $O$ of a cross section. We thus conclude that $O$ is the shear center of the section.

Angle shapes with one vertical and one horizontal leg are encountered in many structures. It follows from the preceding discussion that such members will not be twisted if vertical loads are applied along the center line of their vertical leg. We note from Fig. 6.57 that the resultant of the elementary shearing forces exerted on the vertical portion OA of a given section will be equal to the
shear \( V \), while the resultant of the shearing forces on the horizontal portion \( OB \) will be zero:

\[
\int_{O}^{A} q \, ds = V \quad \text{and} \quad \int_{O}^{B} q \, ds = 0
\]

This does not mean, however, that there will be no shearing stress in the horizontal leg of the member. By resolving the shear \( V \) into components perpendicular to the principal centroidal axes of the section and computing the shearing stress at every point, we would verify that \( \tau \) is zero at only one point between \( O \) and \( B \) (see Sample Prob. 6.6).

Another type of thin-walled member frequently encountered in practice is the \( Z \) shape. While the cross section of a \( Z \) shape does not possess any axis of symmetry, it does possess a center of symmetry \( O \) (Fig. 6.58). This means that, to any point \( H \) of the cross section corresponds another point \( H' \) such that the segment of straight line \( HH' \) is bisected by \( O \). Clearly, the center of symmetry \( O \) coincides with the centroid of the cross section. As you will see presently, point \( O \) is also the shear center of the cross section.

As we did earlier in the case of an angle shape, we assume that the loads are applied in a plane perpendicular to one of the principal axes of the section, so that this axis is also the neutral axis of the section (Fig. 6.59). We further assume that the shearing stresses in the section are defined by Eq. (6.6), i.e., that the member is bent without being twisted. Denoting by \( Q \) the first moment about the neutral axis of portion \( AH \) of the cross section, and by \( Q' \) the first moment of portion \( EH' \), we note that \( Q' = -Q \). Thus the shearing stresses at \( H \) and \( H' \) have the same magnitude and the same direction, and the shearing forces exerted on small elements of area \( dA \) located respectively at \( H \) and \( H' \) are equal forces that have equal and opposite moments about \( O \) (Fig. 6.60). Since this is true for any pair of symmetric elements, it follows that the resultant of the shearing forces exerted on the section has a zero moment about \( O \). This means that the shear \( V \) in the section is directed along a line that passes through \( O \). Since this analysis can be repeated when the loads are applied in a plane perpendicular to the other principal axis, we conclude that point \( O \) is the shear center of the section.
SAMPLE PROBLEM 6.6

Determine the distribution of shearing stresses in the thin-walled angle shape DE of uniform thickness \( t \) for the loading shown.

SOLUTION

Shear Center. We recall from Sec. 6.9 that the shear center of the cross section of a thin-walled angle shape is located at its corner. Since the load \( P \) is applied at \( D \), it causes bending but no twisting of the shape.

Principal Axes. We locate the centroid \( C \) of a given cross section \( AOB \). Since the \( y' \) axis is an axis of symmetry, the \( y' \) and \( z' \) axes are the principal centroidal axes of the section. We recall that for the parallelogram shown, we now determine the centroidal moments of inertia in the section is equal to the load \( P \). We resolve it into components parallel to the principal axes.

Shearing Stresses Due to \( V_y' \). We determine the shearing stress at point \( e \) of coordinate \( y \):

\[
\tau_e = \frac{V_y' y}{I_x t} = \frac{P y \cos 45^\circ}{\frac{1}{12} t (a y - y) t \cos 45^\circ} = \frac{3P(a - y) y}{t^3}
\]

The shearing stress at point \( f \) is represented by a similar function of \( z \).

Shearing Stresses Due to \( V_z' \). We again consider point \( e \):

\[
\tau_e = \frac{V_z' y}{I_y t} = \frac{P z \cos 45^\circ}{\frac{1}{12} t (a z - z) t \cos 45^\circ} = \frac{3P(a^2 - y^2)}{4t^3}
\]

The shearing stress at point \( f \) is represented by a similar function of \( z \).

Combined Stresses. Along the Vertical Leg. The shearing stress at point \( e \) is

\[
\tau_e = \tau_2 + \tau_1 = \frac{3P(a^2 - y^2)}{4t^3} + \frac{3P(a - y) y}{t^3} = \frac{3P(a - y)(a + y + 4y)}{4t^3}
\]

\[
\tau_e = \frac{3P(a - y)(a + 5y)}{4t^3}
\]

Along the Horizontal Leg. The shearing stress at point \( f \) is

\[
\tau_f = \tau_2 - \tau_1 = \frac{3P(a^2 - z^2)}{4t^3} - \frac{3P(a - z) z}{t^3} = \frac{3P(a - z)(a + z - 4z)}{4t^3}
\]

\[
\tau_f = \frac{3P(a - z)(a - 3z)}{4t^3}
\]
6.61 and 6.62 Determine the location of the shear center $O$ of a thin-walled beam of uniform thickness having the cross section shown.

6.63 through 6.66 An extruded beam has the cross section shown. Determine (a) the location of the shear center $O$, (b) the distribution of the shearing stresses caused by the vertical shearing force $V$ shown applied at $O$. 
6.67 through 6.68  An extruded beam has the cross section shown. Determine (a) the location of the shear center $O$, (b) the distribution of the shearing stresses caused by the vertical shearing force $V$ shown applied at $O$.

![Fig. P6.67](Image)

![Fig. P6.68](Image)

6.69 through 6.74  Determine the location of the shear center $O$ of a thin-walled beam of uniform thickness having the cross section shown.

![Fig. P6.69](Image)

![Fig. P6.70](Image)

![Fig. P6.71](Image)

![Fig. P6.72](Image)

![Fig. P6.73](Image)

![Fig. P6.74](Image)
6.75 and 6.76 A thin-walled beam has the cross section shown. Determine the location of the shear center $O$ of the cross section.

![Fig. P6.75](image1)

![Fig. P6.76](image2)

6.77 and 6.78 A thin-walled beam of uniform thickness has the cross section shown. Determine the dimension $b$ for which the shear center $O$ of the cross section is located at the point indicated.

![Fig. P6.77](image3)

![Fig. P6.78](image4)

6.79 For the angle shape and loading of Sample Prob. 6.6, check that $\int q \, dz = 0$ along the horizontal leg of the angle and $\int q \, dy = P$ along its vertical leg.

6.80 For the angle shape and loading of Sample Prob. 6.6, (a) determine the points where the shearing stress is maximum and the corresponding values of the stress, (b) verify that the points obtained are located on the neutral axis corresponding to the given loading.
426 Shearing Stresses in Beams and Thin-Walled Members

*6.81 A steel plate, 160 mm wide and 8 mm thick, is bent to form the channel shown. Knowing that the vertical load \( P \) acts at a point in the midplane of the web of the channel, determine (a) the torque \( T \) that would cause the channel to twist in the same way that it does under the load \( P \), (b) the maximum shearing stress in the channel caused by the load \( P \).

*6.82 Solve Prob. 6.81, assuming that a 6-mm-thick plate is bent to form the channel shown.

*6.83 The cantilever beam \( AB \), consisting of half of a thin-walled pipe of 1.25-in. mean radius and 3/8-in. wall thickness, is subjected to a 500-lb vertical load. Knowing that the line of action of the load passes through the centroid \( C \) of the cross section of the beam, determine (a) the equivalent force-couple system at the shear center of the cross section, (b) the maximum shearing stress in the beam. (Hint: The shear center \( O \) of this cross section was shown in Prob. 6.73 to be located twice as far from its vertical diameter as its centroid \( C \).)

*6.84 Solve Prob. 6.83, assuming that the thickness of the beam is reduced to 1/4 in.

*6.85 The cantilever beam shown consists of a Z shape of 1/4-in. thickness. For the given loading, determine the distribution of the shearing stresses along line \( A'B' \) in the upper horizontal leg of the Z shape. The \( x' \) and \( y' \) axes are the principal centroidal axes of the cross section and the corresponding moments of inertia are \( I_{x'} = 166.3 \text{ in}^4 \) and \( I_{y'} = 138.5 \text{ in}^4 \).

*6.86 For the cantilever beam and loading of Prob. 6.85, determine the distribution of the shearing stress along line \( B'D' \) in the vertical web of the Z shape.

*6.87 Determine the distribution of the shearing stresses along line \( D'B' \) in the horizontal leg of the angle shape for the loading shown. The \( x' \) and \( y' \) axes are the principal centroidal axes of the cross section.

*6.88 For the angle shape and loading of Prob. 6.87, determine the distribution of the shearing stresses along line \( D'A' \) in the vertical leg.
This chapter was devoted to the analysis of beams and thin-walled members under transverse loadings.

In Sec. 6.1 we considered a small element located in the vertical plane of symmetry of a beam under a transverse loading (Fig. 6.61) and found that normal stresses $\sigma_x$ and shearing stresses $\tau_{xy}$ were exerted on the transverse faces of that element, while shearing stresses $\tau_{yx}$, equal in magnitude to $\tau_{xy}$, were exerted on its horizontal faces.

In Sec. 6.2 we considered a prismatic beam $AB$ with a vertical plane of symmetry supporting various concentrated and distributed loads (Fig. 6.62). At a distance $x$ from end $A$ we detached from the beam an element $CDD'C'$ of length $\Delta x$ extending across the width of the beam from the upper surface of the beam to a horizontal plane located at a distance $y_1$ from the neutral axis (Fig. 6.63). We found that the magnitude of the shearing force $\Delta H$ exerted on the lower face of the beam element was

$$\Delta H = \frac{VQ}{I} \Delta x \quad (6.4)$$

where $V = \text{vertical shear in the given transverse section}$

$Q = \text{first moment with respect to the neutral axis of the shaded portion of the section}$

$I = \text{centroidal moment of inertia of the entire cross-sectional area}$
The horizontal shear per unit length, or shear flow, which was denoted by the letter $q$, was obtained by dividing both members of Eq. (6.4) by $\Delta x$:

$$q = \frac{\Delta H}{\Delta x} = \frac{VQ}{I}$$  \hspace{1cm} (6.5)

Dividing both members of Eq. (6.4) by the area $\Delta A$ of the horizontal face of the element and observing that $\Delta A = t \Delta x$, where $t$ is the width of the element at the cut, we obtained in Sec. 6.3 the following expression for the average shearing stress on the horizontal face of the element

$$\tau_{ave} = \frac{VQ}{It}$$  \hspace{1cm} (6.6)

We further noted that, since the shearing stresses $\tau_{xy}$ and $\tau_{yx}$ exerted, respectively, on a transverse and a horizontal plane through $D'$ are equal, the expression in (6.6) also represents the average value of $\tau_{xy}$ along the line $D'D''$ (Fig. 6.64).

In Secs. 6.4 and 6.5 we analyzed the shearing stresses in a beam of rectangular cross section. We found that the distribution of stresses is parabolic and that the maximum stress, which occurs at the center of the section, is

$$\tau_{max} = \frac{3V}{2A}$$  \hspace{1cm} (6.10)

where $A$ is the area of the rectangular section. For wide-flange beams, we found that a good approximation of the maximum shearing stress can be obtained by dividing the shear $V$ by the cross-sectional area of the web.

In Sec. 6.6 we showed that Eqs. (6.4) and (6.5) could still be used to determine, respectively, the longitudinal shearing force $\Delta H$ and the shear flow $q$ exerted on a beam element if the element was bounded by an arbitrary curved surface instead of a horizontal plane (Fig. 6.65).
This made it possible for us in Sec. 6.7 to extend the use of Eq. (6.6) to the determination of the average shearing stress in thin-walled members such as wide-flange beams and box beams, in the flanges of such members, and in their webs (Fig. 6.66).

In Sec. 6.8 we considered the effect of plastic deformations on the magnitude and distribution of shearing stresses. From Chap. 4 we recalled that once plastic deformation has been initiated, additional loading causes plastic zones to penetrate into the elastic core of a beam. After demonstrating that shearing stresses can occur only in the elastic core of a beam, we noted that both an increase in loading and the resulting decrease in the size of the elastic core contribute to an increase in shearing stresses.

In Sec. 6.9 we considered prismatic members that are not loaded in their plane of symmetry and observed that, in general, both bending and twisting will occur. You learned to locate the point $O$ of the cross section, known as the shear center, where the loads should be applied if the member is to bend without twisting (Fig. 6.67) and found that if the loads are applied at that point, the following equations remain valid:

$$\sigma_x = -\frac{My}{I}, \quad \tau_{ave} = \frac{VQ}{It} \quad (4.16, 6.6)$$

Using the principle of superposition, you also learned to determine the stresses in unsymmetric thin-walled members such as channels, angles, and extruded beams [Example 6.07 and Sample Prob. 6.6]
6.89 A square box beam is made of two 20 × 80-mm planks and two 20 × 120-mm planks nailed together as shown. Knowing that the spacing between the nails is \( s = 30 \text{ mm} \) and that the vertical shear in the beam is \( V = 1200 \text{ N} \), determine (a) the shearing force in each nail, (b) the maximum shearing stress in the beam.

![Fig. P6.89](image)

6.90 The beam shown is fabricated by connecting two channel shapes and two plates, using bolts of \( \frac{3}{4} \text{ in.} \) diameter spaced longitudinally every 7.5 in. Determine the average shearing stress in the bolts caused by a shearing force of 25 kips parallel to the \( y \) axis.

6.91 For the beam and loading shown, consider section \( n-n \) and determine (a) the largest shearing stress in that section, (b) the shearing stress at point \( a \).

![Fig. P6.91](image)
6.92 For the beam and loading shown, determine the minimum required width $b$, knowing that for the grade of timber used, $\sigma_{\text{all}} = 12$ MPa and $\tau_{\text{all}} = 825$ kPa.

![Fig. P6.92](image)

6.93 For the beam and loading shown, consider section $n-n$ and determine the shearing stress at (a) point $a$, (b) point $b$.

![Fig. P6.93 and P6.94](image)

6.94 For the beam and loading shown, determine the largest shearing stress in section $n-n$.

6.95 The composite beam shown is made by welding C200 $\times$ 17.1 rolled-steel channels to the flanges of a W250 $\times$ 80 wide-flange rolled-steel shape. Knowing that the beam is subjected to a vertical shear of 200 kN, determine (a) the horizontal shearing force per meter at each weld, (b) the shearing stress at point $a$ of the flange of the wide-flange shape.

![Fig. P6.95](image)
6.96 An extruded beam has the cross section shown and a uniform wall thickness of 3 mm. For a vertical shear of 10 kN, determine (a) the shearing stress at point A, (b) the maximum shearing stress in the beam. Also sketch the shear flow in the cross section.

![Fig. P6.96](image)

6.97 The design of a beam requires welding four horizontal plates to a vertical 0.5 × 5-in. plate as shown. For a vertical shear \( V \), determine the dimension \( h \) for which the shear flow through the welded surfaces is maximum.

![Fig. P6.97](image)

6.98 Determine the location of the shear center \( O \) of a thin-walled beam of uniform thickness having the cross section shown.

![Fig. P6.98](image)
6.99 Determine the location of the shear center $O$ of a thin-walled beam of uniform thickness having the cross section shown.

![Fig. P6.99](image)

6.100 A thin-walled beam of uniform thickness has the cross section shown. Determine the dimension $b$ for which the shear center $O$ of the cross section is located at the point indicated.

![Fig. P6.100](image)
COMPUTER PROBLEMS

The following problems are designed to be solved with a computer.

6.C1 A timber beam is to be designed to support a distributed load and up to two concentrated loads as shown. One of the dimensions of its uniform rectangular cross section has been specified and the other is to be determined so that the maximum normal stress and the maximum shearing stress in the beam will not exceed given allowable values $\sigma_{\text{all}}$ and $\tau_{\text{all}}$. Measuring $x$ from end A and using either SI or U.S. customary units, write a computer program to calculate for successive cross sections, from $x = 0$ to $x = L$ and using given increments $\Delta x$, the shear, the bending moment, and the smallest value of the unknown dimension that satisfies in that section (1) the allowable normal stress requirement, (2) the allowable shearing stress requirement. Use this program to solve Prob. 5.65 assuming $\sigma_{\text{all}} = 12$ MPa and $\tau_{\text{all}} = 825$ kPa, using $\Delta x = 0.1$ m.

6.C2 A cantilever timber beam $AB$ of length $L$ and of uniform rectangular section shown supports a concentrated load $P$ at its free end and a uniformly distributed load $w$ along its entire length. Write a computer program to determine the length $L$ and the width $b$ of the beam for which both the maximum normal stress and the maximum shearing stress in the beam reach their largest allowable values. Assuming $\sigma_{\text{all}} = 1.8$ ksi and $\tau_{\text{all}} = 120$ psi, use this program to determine the dimensions $L$ and $b$ when (a) $P = 1000$ lb and $w = 0$, (b) $P = 0$ and $w = 12.5$ lb/in., (c) $P = 500$ lb and $w = 12.5$ lb/in.

6.C3 A beam having the cross section shown is subjected to a vertical shear $V$. Write a computer program that, for loads and dimensions expressed in either SI or U.S. customary units, can be used to calculate the shearing stress along the line between any two adjacent rectangular areas forming the cross section. Use this program to solve (a) Prob. 6.10, (b) Prob. 6.12, (c) Prob. 6.21.
6.C4 A plate of uniform thickness $t$ is bent as shown into a shape with a vertical plane of symmetry and is then used as a beam. Write a computer program that, for loads and dimensions expressed in either SI or U.S. customary units, can be used to determine the distribution of shearing stresses caused by a vertical shear $V$. Use this program (a) to solve Prob. 6.47, (b) to find the shearing stress at a point $E$ for the shape and load of Prob. 6.50, assuming a thickness $t = \frac{1}{2}$ in.

Fig. P6.C4

6.C5 The cross section of an extruded beam is symmetric with respect to the $x$ axis and consists of several straight segments as shown. Write a computer program that, for loads and dimensions expressed in either SI or U.S. customary units, can be used to determine (a) the location of the shear center $O$, (b) the distribution of shearing stresses caused by a vertical force applied at $O$. Use this program to solve Probs. 6.66 and 6.70.

Fig. P6.C5

6.C6 A thin-walled beam has the cross section shown. Write a computer program that, for loads and dimensions expressed in either SI or U.S. customary units, can be used to determine the location of the shear center $O$ of the cross section. Use the program to solve Prob. 6.75.

Fig. P6.C6
The aircraft shown is being tested to determine how the forces due to lift would be distributed over the wing. This chapter deals with stresses and strains in structures and machine components.
CHAPTER 7

Transformations of Stress and Strain

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Chapter 7 Transformations of Stress and Strain

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7.1 INTRODUCTION

We saw in Sec. 1.12 that the most general state of stress at a given point \( Q \) may be represented by six components. Three of these components, \( \sigma_x, \sigma_y, \) and \( \sigma_z \), define the normal stresses exerted on the faces of a small cubic element centered at \( Q \) and of the same orientation as the coordinate axes (Fig. 7.1a), and the other three, \( \tau_{yx}, \tau_{zy}, \) and \( \tau_{xz}, \)† the components of the shearing stresses on the same element. As we remarked at the time, the same state of stress will be represented by a different set of components if the coordinate axes are rotated (Fig. 7.1b). We propose in the first part of this chapter to determine how the components of stress are transformed under a rotation of the coordinate axes. The second part of the chapter will be devoted to a similar analysis of the transformation of the components of strain.

†We recall that \( \tau_{xy} = \tau_{yx}, \tau_{yz} = \tau_{zy}, \) and \( \tau_{zx} = \tau_{xz}. \)

Our discussion of the transformation of stress will deal mainly with plane stress, i.e., with a situation in which two of the faces of the cubic element are free of any stress. If the \( z \) axis is chosen perpendicular to these faces, we have \( \sigma_z = \tau_{zx} = \tau_{zy} = 0, \) and the only remaining stress components are \( \sigma_x, \sigma_y, \) and \( \tau_{xy} \) (Fig. 7.2). Such a situation occurs in a thin plate subjected to forces acting in the mid-plane of the plate (Fig. 7.3). It also occurs on the free surface of a structural element or machine component, i.e., at any point of the surface of that element or component that is not subjected to an external force (Fig. 7.4).

Fig. 7.1 General state of stress at a point.

Fig. 7.2 Plane stress.

Fig. 7.3 Example of plane stress.

Fig. 7.4 Example of plane stress.
Considering in Sec. 7.2 a state of plane stress at a given point \( Q \) characterized by the stress components \( \sigma_x, \sigma_y, \) and \( \tau_{xy} \) associated with the element shown in Fig. 7.5(a), you will learn to determine the components \( \sigma_{x'}, \sigma_{y'}, \) and \( \tau_{x'y'} \) associated with that element after it has been rotated through an angle \( \theta \) about the \( z \) axis (Fig. 7.5(b)). In Sec. 7.3, you will determine the value \( \theta_p \) of \( \theta \) for which the stresses \( \sigma_x \) and \( \sigma_y \) are, respectively, maximum and minimum; these values of the normal stress are the principal stresses at point \( Q \), and the faces of the corresponding element define the principal planes of stress at that point. You will also determine the value \( \theta_s \) of the angle of rotation for which the shearing stress is maximum, as well as the value of that stress.

**Fig. 7.5** Transformation of stress

In Sec. 7.4, an alternative method for the solution of problems involving the transformation of plane stress, based on the use of Mohr’s circle, will be presented.

In Sec. 7.5, the three-dimensional state of stress at a given point will be considered and a formula for the determination of the normal stress on a plane of arbitrary orientation at that point will be developed. In Sec. 7.6, you will consider the rotations of a cubic element about each of the principal axes of stress and note that the corresponding transformations of stress can be described by three different Mohr’s circles. You will also observe that, in the case of a state of plane stress at a given point, the maximum value of the shearing stress obtained earlier by considering rotations in the plane of stress does not necessarily represent the maximum shearing stress at that point. This will bring you to distinguish between in-plane and out-of-plane maximum shearing stresses.

**Yield criteria** for ductile materials under plane stress will be developed in Sec. 7.7. To predict whether a material will yield at some critical point under given loading conditions, you will determine the principal stresses \( \sigma_a \) and \( \sigma_b \) at that point and check whether \( \sigma_a, \sigma_b, \) and the yield strength \( \sigma_y \) of the material satisfy some criterion. Two criteria in common use are: the maximum-shearing-strength criterion and the maximum-distortion-energy criterion. In Sec. 7.8, **fracture criteria** for brittle materials under plane stress will be developed in a similar fashion; they will involve the principal stresses \( \sigma_a \) and \( \sigma_b \) at some critical point and the ultimate strength \( \sigma_U \) of the
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Material. Two criteria will be discussed: the maximum-normal-stress criterion and Mohr's criterion.

Thin-walled pressure vessels provide an important application of the analysis of plane stress. In Sec. 7.9, we will discuss stresses in both cylindrical and spherical pressure vessels (Photos 7.1 and 7.2).

Sections 7.10 and 7.11 will be devoted to a discussion of the transformation of plane strain and to Mohr's circle for plane strain. In Sec. 7.12, we will consider the three-dimensional analysis of strain and see how Mohr's circles can be used to determine the maximum shearing strain at a given point. Two particular cases are of special interest and should not be confused: the case of plane strain and the case of plane stress.

Finally, in Sec. 7.13, we discuss the use of strain gages to measure the normal strain on the surface of a structural element or machine component. You will see how the components $e_x$, $e_y$, and $\gamma_{xy}$ characterizing the state of strain at a given point can be computed from the measurements made with three strain gages forming a strain rosette.

7.2 TRANSFORMATION OF PLANE STRESS

Let us assume that a state of plane stress exists at point $Q$ (with $\sigma_z = \tau_{zx} = \tau_{zy} = 0$), and that it is defined by the stress components $\sigma_x$, $\sigma_y$, and $\tau_{xy}$ associated with the element shown in Fig. 7.5a. We propose to determine the stress components $\sigma_{x'}$, $\sigma_{y'}$, and $\tau_{x'y'}$ associated with the element after it has been rotated through an angle $\theta$ about
the $z$ axis (Fig. 7.5b), and to express these components in terms of
$s_x$, $s_y$, $t_{xy}$, and $\theta$.

In order to determine the normal stress $s_x$ and the shearing stress $t_{xy}$ exerted on the face perpendicular to the $x'$ axis, we con-
sider a prismatic element with faces respectively perpendicular to
the $x$, $y$, and $x'$ axes (Fig. 7.6a). We observe that, if the area of the
oblique face is denoted by $\Delta A$, the areas of the vertical and horizon-
tal faces are respectively equal to $\Delta A \cos \theta$ and $\Delta A \sin \theta$. It follows that the forces exerted on the three faces are as shown in Fig. 7.6b.
(No forces are exerted on the triangular faces of the element, since
the corresponding normal and shearing stresses have all been assumed
equal to zero.) Using components along the $x'$ and $y'$ axes, we write
the following equilibrium equations:

\[ \sum F_x = 0: \quad s_x \Delta A - s_y(\Delta A \cos \theta) \cos \theta - t_{xy}(\Delta A \cos \theta) \sin \theta - s_y(\Delta A \sin \theta) \sin \theta - t_{xy}(\Delta A \sin \theta) \cos \theta = 0 \]

\[ \sum F_{y'} = 0: \quad t_{x'y'} \Delta A + s_x(\Delta A \cos \theta) \sin \theta - t_{xy}(\Delta A \cos \theta) \cos \theta - s_y(\Delta A \sin \theta) \cos \theta + t_{xy}(\Delta A \sin \theta) \sin \theta = 0 \]
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Solving the first equation for $s_x$ and the second for $t_x$, we have

$$
\sigma_{x'} = \sigma_x \cos^2 \theta + \sigma_y \sin^2 \theta + 2 \tau_{xy} \sin \theta \cos \theta \quad (7.1)
$$

$$
\tau_{x'y'} = -(\sigma_x - \sigma_y) \sin \theta \cos \theta + \tau_{xy} (\cos^2 \theta - \sin^2 \theta) \quad (7.2)
$$

Recalling the trigonometric relations

$$
\sin 2\theta = 2 \sin \theta \cos \theta \quad \cos 2\theta = \cos^2 \theta - \sin^2 \theta \quad (7.3)
$$

and

$$
\cos^2 \theta = \frac{1 + \cos 2\theta}{2} \quad \sin^2 \theta = \frac{1 - \cos 2\theta}{2} \quad (7.4)
$$

we write Eq. (7.1) as follows:

$$
\sigma_{x'} = \sigma_x \frac{1 + \cos 2\theta}{2} + \sigma_y \frac{1 - \cos 2\theta}{2} + \tau_{xy} \sin 2\theta
$$

or

$$
\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \quad (7.5)
$$

Using the relations (7.3), we write Eq. (7.2) as

$$
\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta \quad (7.6)
$$

The expression for the normal stress $\sigma_{y'}$ is obtained by replacing $\theta$ in Eq. (7.5) by the angle $\theta + 90^\circ$ that the $y'$ axis forms with the $x$ axis. Since $\cos (2\theta + 180^\circ) = -\cos 2\theta$ and $\sin (2\theta + 180^\circ) = -\sin 2\theta$, we have

$$
\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta \quad (7.7)
$$

Adding Eqs. (7.5) and (7.7) member to member, we obtain

$$
\sigma_{x'} + \sigma_{y'} = \sigma_x + \sigma_y \quad (7.8)
$$

Since $\sigma_z = \sigma_z = 0$, we thus verify in the case of plane stress that the sum of the normal stresses exerted on a cubic element of material is independent of the orientation of that element.†

†Cf. first footnote on page 97.
7.3 PRINCIPAL STRESSES; MAXIMUM SHEARING STRESS

The equations (7.5) and (7.6) obtained in the preceding section are the parametric equations of a circle. This means that, if we choose a set of rectangular axes and plot a point $M$ of abscissa $\sigma_x$ and ordinate $\tau_{xy}$ for any given value of the parameter $\theta$, all the points thus obtained will lie on a circle. To establish this property we eliminate $\theta$ from Eqs. (7.5) and (7.6); this is done by first transposing $(\sigma_x + \sigma_y)/2$ in Eq. (7.5) and squaring both members of the equation, then squaring both members of Eq. (7.6), and finally adding member to member the two equations obtained in this fashion. We have

$$\left(\sigma_x' - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{xy}'^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$  \hspace{1cm} (7.9)

Setting

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} \quad \text{and} \quad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$  \hspace{1cm} (7.10)

we write the identity (7.9) in the form

$$(\sigma_x' - \sigma_{ave})^2 + \tau_{xy}'^2 = R^2$$  \hspace{1cm} (7.11)

which is the equation of a circle of radius $R$ centered at the point $C$ of abscissa $\sigma_{ave}$ and ordinate $-\tau_{xy}'$. It will be observed that, due to the symmetry of the circle about the horizontal axis, the same result would have been obtained if, instead of plotting $M$, we had plotted a point $N$ of abscissa $\sigma_x'$ and ordinate $-\tau_{xy}'$ (Fig. 7.8). This property will be used in Sec. 7.4.

The two points $A$ and $B$ where the circle of Fig. 7.7 intersects the horizontal axis are of special interest: Point $A$ corresponds to the maximum value of the normal stress $\sigma_x'$, while point $B$ corresponds
to its minimum value. Besides, both points correspond to a zero value of the shearing stress \( \tau_{xy} \). Thus, the values \( \theta_p \) of the parameter \( \theta \) which correspond to points \( A \) and \( B \) can be obtained by setting \( \tau_{xy} = 0 \) in Eq. (7.6). We write‡

\[
\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}
\]

(7.12)

This equation defines two values \( 2\theta_p \) that are 180° apart, and thus two values \( \theta_p \) that are 90° apart. Either of these values can be used to determine the orientation of the corresponding element (Fig. 7.9).

**Fig. 7.9** Principal stresses.

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The planes containing the faces of the element obtained in this way are called the *principal planes of stress* at point \( Q \), and the corresponding values \( \sigma_{\text{max}} \) and \( \sigma_{\text{min}} \) of the normal stress exerted on these planes are called the *principal stresses* at \( Q \). Since the two values \( \theta_p \) defined by Eq. (7.12) were obtained by setting \( \tau_{xy} = 0 \) in Eq. (7.6), it is clear that no shearing stress is exerted on the principal planes.

We observe from Fig. 7.7 that

\[
\sigma_{\text{max}} = \sigma_{\text{ave}} + R \quad \text{and} \quad \sigma_{\text{min}} = \sigma_{\text{ave}} - R
\]

(7.13)

Substituting for \( \sigma_{\text{ave}} \) and \( R \) from Eq. (7.10), we write

\[
\sigma_{\text{max}, \text{min}} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}
\]

(7.14)

Unless it is possible to tell by inspection which of the two principal planes is subjected to \( \sigma_{\text{max}} \) and which is subjected to \( \sigma_{\text{min}} \), it is necessary to substitute one of the values \( \theta_p \) into Eq. (7.5) in order to determine which of the two corresponds to the maximum value of the normal stress.

Referring again to the circle of Fig. 7.7, we note that the points \( D \) and \( E \) located on the vertical diameter of the circle correspond to

‡This relation can also be obtained by differentiating \( \sigma_x \) in Eq. (7.5) and setting the derivative equal to zero: \( d\sigma_x/d\theta = 0 \).
the largest numerical value of the shearing stress $\tau_{xy}$. Since the abscissa of points $D$ and $E$ is $s_{xy} = (s_x + s_y)/2$, the values $\theta_1$ of the parameter $\theta$ corresponding to these points are obtained by setting $\sigma_z = (\sigma_x + \sigma_y)/2$ in Eq. (7.5). It follows that the sum of the last two terms in that equation must be zero. Thus, for $\theta = \theta_1$, we write†

$$\frac{\sigma_x - \sigma_y}{2} \cos 2\theta_x + \tau_{xy} \sin 2\theta_x = 0$$

or

$$\tan 2\theta_x = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$

(7.15)

This equation defines two values $2\theta_1$ that are $180^\circ$ apart, and thus two values $\theta_1$ that are $90^\circ$ apart. Either of these values can be used to determine the orientation of the element corresponding to the maximum shearing stress (Fig. 7.10). Observing from Fig. 7.7 that the maximum value of the shearing stress is equal to the radius $R$ of the circle, and recalling the second of Eqs. (7.10), we write

$$\tau_{\text{max}} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

(7.16)

As observed earlier, the normal stress corresponding to the condition of maximum shearing stress is

$$\sigma' = \sigma_{\text{ave}} = \frac{\sigma_x + \sigma_y}{2}$$

(7.17)

Comparing Eqs. (7.12) and (7.15), we note that $\tan 2\theta_1$ is the negative reciprocal of $\tan 2\theta_x$. This means that the angles $2\theta_1$ and $2\theta_x$ are $90^\circ$ apart and, therefore, that the angles $\theta_1$ and $\theta_x$ are $45^\circ$ apart. We thus conclude that the planes of maximum shearing stress are at $45^\circ$ to the principal planes. This confirms the results obtained earlier in Sec. 1.12 in the case of a centric axial loading (Fig. 1.38) and in Sec. 3.4 in the case of a torsional loading (Fig. 3.19.)

We should be aware that our analysis of the transformation of plane stress has been limited to rotations in the plane of stress. If the cubic element of Fig. 7.5 is rotated about an axis other than the $z$ axis, its faces may be subjected to shearing stresses larger than the stress defined by Eq. (7.16). As you will see in Sec. 7.5, this occurs when the principal stresses defined by Eq. (7.14) have the same sign, i.e., when they are either both tensile or both compressive. In such cases, the value given by Eq. (7.16) is referred to as the maximum in-plane shearing stress.

†This relation may also be obtained by differentiating $\tau_{xy}$ in Eq. (7.6) and setting the derivative equal to zero: $d\tau_{xy}/d\theta = 0$. 

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For the state of plane stress shown in Fig. 7.11, determine (a) the principal planes, (b) the principal stresses, (c) the maximum shearing stress and the corresponding normal stress.

(a) Principal Planes. Following the usual sign convention, we write the stress components as

\[
\sigma_x = +50 \text{ MPa} \quad \sigma_y = -10 \text{ MPa} \quad \tau_{xy} = +40 \text{ MPa}
\]

Substituting into Eq. (7.12), we have

\[
\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(40)}{50 - (-10)} = \frac{80}{60} = 1.33
\]

\[
2\theta_p = 53.1^\circ \quad \text{and} \quad 180^\circ + 53.1^\circ = 233.1^\circ
\]

\[
\theta_p = 26.6^\circ \quad \text{and} \quad 116.6^\circ
\]

(b) Principal Stresses. Formula (7.14) yields

\[
\sigma_{max, min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}
\]

\[
\sigma_{max} = 20 + 50 = 70 \text{ MPa}
\]

\[
\sigma_{min} = 20 - 50 = -30 \text{ MPa}
\]

The principal planes and principal stresses are sketched in Fig. 7.12. Making \(\theta = 26.6^\circ\) in Eq. (7.5), we check that the normal stress exerted on face BC of the element is the maximum stress:

\[
\sigma_x = \frac{50 - 10}{2} + \frac{50 + 10}{2} \cos 53.1^\circ + 40 \sin 53.1^\circ
\]

\[
= 20 + 30 \cos 53.1^\circ + 40 \sin 53.1^\circ = 70 \text{ MPa} = \sigma_{max}
\]

(c) Maximum Shearing Stress. Formula (7.16) yields

\[
\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{(30)^2 + (40)^2} = 50 \text{ MPa}
\]

Since \(\sigma_{max}\) and \(\sigma_{min}\) have opposite signs, the value obtained for \(\tau_{max}\) actually represents the maximum value of the shearing stress at the point considered. The orientation of the planes of maximum shearing stress and the sense of the shearing stresses are best determined by passing a section along the diagonal plane AC of the element of Fig. 7.12. Since the faces AB and BC of the element are contained in the principal planes, the diagonal plane AC must be one of the planes of maximum shear stress (Fig. 7.13). Furthermore, the equilibrium conditions for the prismatic element ABC require that the shearing stress exerted on AC be directed as shown. The cubic element corresponding to the maximum shearing stress is shown in Fig. 7.14. The normal stress on each of the four faces of the element is given by Eq. (7.17):

\[
\sigma' = \sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = \frac{50 - 10}{2} = 20 \text{ MPa}
\]
SAMPLE PROBLEM 7.1

A single horizontal force $\mathbf{P}$ of magnitude 150 lb is applied to end $D$ of lever $ABD$. Knowing that portion $AB$ of the lever has a diameter of 1.2 in., determine (a) the normal and shearing stresses on an element located at point $H$ and having sides parallel to the $x$ and $y$ axes, (b) the principal planes and the principal stresses at point $H$.

SOLUTION

**Force-Couple System.** We replace the force $\mathbf{P}$ by an equivalent force-couple system at the center $C$ of the transverse section containing point $H$:

\[
\begin{align*}
\mathbf{P} &= 150 \text{ lb} \\
T &= (150 \text{ lb})(18 \text{ in.}) = 2.7 \text{ kip} \cdot \text{in.} \\
M_x &= (150 \text{ lb})(10 \text{ in.}) = 1.5 \text{ kip} \cdot \text{in.}
\end{align*}
\]

**a. Stresses $\sigma_x$, $\sigma_y$, $\tau_{xy}$ at Point H.** Using the sign convention shown in Fig. 7.2, we determine the sense and the sign of each stress component by carefully examining the sketch of the force-couple system at point $C$:

\[
\sigma_x = 0 \\
\sigma_y = + \frac{M_c}{I} = + \frac{(1.5 \text{ kip} \cdot \text{in.})(0.6 \text{ in.})}{\frac{1}{12} \pi (0.6 \text{ in.})^4} = 8.84 \text{ ksi} \\
\tau_{xy} = + \frac{T_c}{J} = + \frac{(2.7 \text{ kip} \cdot \text{in.})(0.6 \text{ in.})}{\frac{1}{12} \pi (0.6 \text{ in.})^4} = 7.96 \text{ ksi}
\]

We note that the shearing force $\mathbf{P}$ does not cause any shearing stress at point $H$.

**b. Principal Planes and Principal Stresses.** Substituting the values of the stress components into Eq. (7.12), we determine the orientation of the principal planes:

\[
\begin{align*}
\tan 2\theta_p &= \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(7.96)}{0 - 8.84} = -1.80 \\
2\theta_p &= -61.0^\circ \quad \text{and} \quad 180^\circ - 61.0^\circ = 119^\circ \\
\theta_p &= -30.5^\circ \quad \text{and} \quad +59.5^\circ
\end{align*}
\]

Substituting into Eq. (7.14), we determine the magnitudes of the principal stresses:

\[
\sigma_{\max, \min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}
\]

\[
= \frac{0 + 8.84}{2} \pm \sqrt{\left(\frac{0 - 8.84}{2}\right)^2 + (7.96)^2} = \pm 9.10
\]

\[
\sigma_{\max} = +13.52 \text{ ksi} \\
\sigma_{\min} = -4.68 \text{ ksi}
\]

Considering face $ab$ of the element shown, we make $\theta_p = -30.5^\circ$ in Eq. (7.5) and find $\sigma_{\max} = -13.52 \text{ ksi}$ and $\sigma_{\min} = -4.68 \text{ ksi}$. We conclude that the principal stresses are as shown.
**PROBLEMS**

**7.1 through 7.4** For the given state of stress, determine the normal and shearing stresses exerted on the oblique face of the shaded triangular element shown. Use a method of analysis based on the equilibrium of that element, as was done in the derivations of Sec. 7.2.

![Fig. P7.1](http://www.opoosoft.com)

![Fig. P7.2](http://www.opoosoft.com)

![Fig. P7.3](http://www.opoosoft.com)

![Fig. P7.4](http://www.opoosoft.com)

**7.5 through 7.8** For the given state of stress, determine (a) the principal planes, (b) the principal stresses.

**7.9 through 7.12** For the given state of stress, determine (a) the orientation of the planes of maximum in-plane shearing stress, (b) the maximum in-plane shearing stress, (c) the corresponding normal stress.

![Fig. P7.5 and P7.9](http://www.opoosoft.com)

![Fig. P7.6 and P7.10](http://www.opoosoft.com)

![Fig. P7.7 and P7.11](http://www.opoosoft.com)

![Fig. P7.8 and P7.12](http://www.opoosoft.com)

**7.13 through 7.16** For the given state of stress, determine the normal and shearing stresses after the element shown has been rotated through (a) 25° clockwise, (b) 10° counterclockwise.

![Fig. P7.13](http://www.opoosoft.com)

![Fig. P7.14](http://www.opoosoft.com)

![Fig. P7.15](http://www.opoosoft.com)

![Fig. P7.16](http://www.opoosoft.com)
7.17 and 7.18 The grain of a wooden member forms an angle of 15° with the vertical. For the state of stress shown, determine (a) the in-plane shearing stress parallel to the grain, (b) the normal stress perpendicular to the grain.

![Fig. P7.17](image1)

![Fig. P7.18](image2)

Fig. P7.17

Fig. P7.18

7.19 A steel pipe of 12-in. outer diameter is fabricated from \( \frac{1}{4} \)-in.-thick plate by welding along a helix that forms an angle of 22.5° with a plane perpendicular to the axis of the pipe. Knowing that a 40-kip axial force \( P \) and an 80-kip \( \cdot \) in. torque \( T \), each directed as shown, are applied to the pipe, determine \( \sigma \) and \( \tau \) in directions, respectively, normal and tangential to the weld.

![Fig. P7.19](image3)

![Fig. P7.20](image4)

Fig. P7.19

Fig. P7.20

7.20 Two members of uniform cross section 50 \( \times \) 80 mm are glued together along plane \( a-a \) that forms an angle of 25° with the horizontal. Knowing that the allowable stresses for the glued joint are \( \sigma = 800 \text{ kPa} \) and \( \tau = 600 \text{ kPa} \), determine the largest centric load \( P \) that can be applied.

7.21 Two steel plates of uniform cross section 10 \( \times \) 80 mm are welded together as shown. Knowing that centric 100-kN forces are applied to the welded plates and that \( \beta = 25° \), determine (a) the in-plane shearing stress parallel to the weld, (b) the normal stress perpendicular to the weld.

![Fig. P7.21 and P7.22](image5)

Fig. P7.21 and P7.22

7.22 Two steel plates of uniform cross section 10 \( \times \) 80 mm are welded together as shown. Knowing that centric 100-kN forces are applied to the welded plates and that the in-plane shearing stress parallel to the weld is 30 MPa, determine (a) the angle \( \beta \), (b) the corresponding normal stress perpendicular to the weld.
7.23 A 400-lb vertical force is applied at D to a gear attached to the solid 1-in. diameter shaft AB. Determine the principal stresses and the maximum shearing stress at point H located as shown on top of the shaft.

7.24 A mechanic uses a crowfoot wrench to loosen a bolt at E. Knowing that the mechanic applies a vertical 24-lb force at A, determine the principal stresses and the maximum shearing stress at point H located as shown on top of the 3/4-in. diameter shaft.

7.25 The steel pipe AB has a 102-mm outer diameter and a 6-mm wall thickness. Knowing that arm CD is rigidly attached to the pipe, determine the principal stresses and the maximum shearing stress at point K.

7.26 The axle of an automobile is acted upon by the forces and couple shown. Knowing that the diameter of the solid axle is 32 mm, determine (a) the principal planes and principal stresses at point H located on top of the axle, (b) the maximum shearing stress at the same point.
For the state of plane stress shown, determine (a) the largest value of \( \tau_{xy} \) for which the maximum in-plane shearing stress is equal to or less than 12 ksi, (b) the corresponding principal stresses.

For the state of plane stress shown, determine the largest value of \( \sigma_y \) for which the maximum in-plane shearing stress is equal to or less than 75 MPa.

Determine the range of values of \( \sigma_x \) for which the maximum in-plane shearing stress is equal to or less than 10 ksi.

For the state of plane stress shown, determine (a) the value of \( \tau_{uy} \) for which the in-plane shearing stress parallel to the weld is zero, (b) the corresponding principal stresses.
7.4 Mohr’s Circle for Plane Stress

The circle used in the preceding section to derive some of the basic formulas relating to the transformation of plane stress was first introduced by the German engineer Otto Mohr (1835–1918) and is known as Mohr’s circle for plane stress. As you will see presently, this circle can be used to obtain an alternative method for the solution of the various problems considered in Secs. 7.2 and 7.3. This method is based on simple geometric considerations and does not require the use of specialized formulas. While originally designed for graphical solutions, it lends itself well to the use of a calculator.

Consider a square element of a material subjected to plane stress (Fig. 7.15a), and let \( \sigma_x, \sigma_y, \) and \( \tau_{xy} \) be the components of the stress exerted on the element. We plot a point \( X \) of coordinates \( \sigma_x \) and \( -\tau_{xy} \), and a point \( Y \) of coordinates \( \sigma_y \) and \( +\tau_{xy} \) (Fig. 7.15b). If \( \tau_{xy} \) is positive, as assumed in Fig. 7.15a, point \( X \) is located below the \( \sigma \) axis and point \( Y \) above, as shown in Fig. 7.15b. If \( \tau_{xy} \) is negative, \( X \) is located above the \( \sigma \) axis and \( Y \) below. Joining \( X \) and \( Y \) by a straight line, we define the point \( C \) of intersection of line \( XY \) with the \( \sigma \) axis and draw the circle of center \( C \) and diameter \( XY \). Noting that the abscissa of \( C \) and the radius of the circle are respectively equal to the quantities \( \sigma_{ave} \) and \( R \) defined by Eqs. (7.10), we conclude that the circle obtained is Mohr’s circle for plane stress. Thus the abscissas of points \( A \) and \( B \) where the circle intersects the \( \sigma \) axis represent respectively the principal stresses \( \sigma_{max} \) and \( \sigma_{min} \) at the point considered.

We also note that, since \( \tan(\angle XCA) = 2\tau_{xy}/(\sigma_x - \sigma_y) \), the angle \( \angle XCA \) is equal in magnitude to one of the angles \( 2\theta_p \) that satisfy Eq. (7.12). Thus, the angle \( \theta_p \) that defines in Fig. 7.15a the orientation of the principal plane corresponding to point \( A \) in Fig. 7.15b can be obtained by dividing in half the angle \( \angle XCA \) measured on Mohr’s circle. We further observe that if \( \sigma_x > \sigma_y \) and \( \tau_{xy} > 0 \), as in the case considered here, the rotation that brings \( CX \) into \( CA \) is counterclockwise. But, in that case, the angle \( \theta_p \) obtained from Eq. (7.12) and defining the direction of the normal \( OA \) to the principal plane is positive; thus, the rotation bringing \( Ox \) into \( Oa \) is also counterclockwise. We conclude that the senses of rotation in both parts of Fig. 7.15 are the same; if a counterclockwise rotation through \( 2\theta_p \) is required to bring \( CX \) into \( CA \) on Mohr’s circle, a counterclockwise rotation through \( \theta_p \) will bring \( Ox \) into \( Oa \) in Fig. 7.15a.†

Since Mohr’s circle is uniquely defined, the same circle can be obtained by considering the stress components \( \sigma_{x'}, \sigma_{y'}, \) and \( \tau_{x'y'} \), corresponding to the \( x' \) and \( y' \) axes shown in Fig. 7.16a. The point \( X' \) of coordinates \( \sigma_{x'} \) and \( -\tau_{x'y'} \), and the point \( Y' \) of coordinates \( \sigma_{y'} \) and \( +\tau_{x'y'} \), are therefore located on Mohr’s circle, and the angle \( X'CA \) in Fig. 7.16b must be equal to twice the angle \( x'Oa \) in Fig. 7.16a. Since, as noted before, the angle \( XCA \) is twice the angle \( zOA \), it follows that

†This is due to the fact that we are using the circle of Fig. 7.8 rather than the circle of Fig. 7.7 as Mohr’s circle.
the angle $XCX'$ in Fig. 7.16b is twice the angle $xOx'$ in Fig. 7.16a. Thus the diameter $XY'$ defining the normal and shearing stresses $\sigma_x', \sigma_y'$, and $\tau_{xy}'$ can be obtained by rotating the diameter $XY$ through an angle equal to twice the angle $\theta$ formed by the $x'$ and $x$ axes in Fig. 7.16a. We note that the rotation that brings the diameter $XY$ into the diameter $X'Y'$ in Fig. 7.16b has the same sense as the rotation that brings the $xy$ axes into the $x'y'$ axes in Fig. 7.16a.

**Fig. 7.16**

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The property we have just indicated can be used to verify the fact that the planes of maximum shearing stress are at $45^\circ$ to the principal planes. Indeed, we recall that points $D$ and $E$ on Mohr's circle correspond to the planes of maximum shearing stress, while $A$ and $B$ correspond to the principal planes (Fig. 7.17b). Since the diameters $AB$ and $DE$ of Mohr's circle are at $90^\circ$ to each other, it follows that the faces of the corresponding elements are at $45^\circ$ to each other (Fig. 7.17a).

**Fig. 7.17**
The construction of Mohr's circle for plane stress is greatly simplified if we consider separately each face of the element used to define the stress components. From Figs. 7.15 and 7.16 we observe that, when the shearing stress exerted on a given face tends to rotate the element clockwise, the point on Mohr's circle corresponding to that face is located above the $\sigma$ axis. When the shearing stress on a given face tends to rotate the element counterclockwise, the point corresponding to that face is located below the $\sigma$ axis (Fig. 7.18).† As far as the normal stresses are concerned, the usual convention holds, i.e., a tensile stress is considered as positive and is plotted to the right, while a compressive stress is considered as negative and is plotted to the left.

†The following jingle is helpful in remembering this convention. “In the kitchen, the clock is above, and the counter is below.”
(b) Principal Planes and Principal Stresses. The principal stresses are

\[ \sigma_{\text{max}} = OA = OC + CA = 20 + 50 = 70 \text{ MPa} \]
\[ \sigma_{\text{min}} = OB = OC - BC = 20 - 50 = -30 \text{ MPa} \]

Recalling that the angle \( ACX \) represents \( 2\theta_p \) (Fig. 7.19b), we write

\[ \tan 2\theta_p = \frac{FX}{CF} = \frac{40}{30} \]
\[ 2\theta_p = 53.1^\circ \quad \theta_p = 26.6^\circ \]

Since the rotation which brings \( CX \) into \( CA \) in Fig. 7.20b is counterclockwise, the rotation that brings \( Ox \) into the axis \( Oa \) corresponding to \( \sigma_{\text{max}} \) in Fig. 7.20a is also counterclockwise.

(c) Maximum Shearing Stress. Since a further rotation of \( 90^\circ \) clockwise brings \( CA \) into \( CD \) in Fig. 7.20b, a further rotation of \( 45^\circ \) counterclockwise will bring the axis \( Oa \) into the axis \( Od \) corresponding to the maximum shearing stress in Fig. 7.20a. We note from Fig. 7.20b that \( \tau_{\text{max}} = R = 50 \text{ MPa} \) and that the corresponding normal stress is \( \sigma' = \sigma_{\text{ave}} = 20 \text{ MPa} \). Since point \( D \) is located above the \( \sigma \) axis in Fig. 7.20b, the shearing stresses exerted on the faces perpendicular to \( Od \) in Fig. 7.20a must be directed so that they will tend to rotate the element clockwise.
Mohr’s circle provides a convenient way of checking the results obtained earlier for stresses under a centric axial loading (Sec. 1.12) and under a torsional loading (Sec. 3.4). In the first case (Fig. 7.21a), we have \( \sigma_x = P/A, \sigma_y = 0, \) and \( \tau_{xy} = 0. \) The corresponding points \( X \) and \( Y \) define a circle of radius \( R = P/2A \) that passes through the origin of coordinates (Fig. 7.21b). Points \( D \) and \( E \) yield the orientation of the planes of maximum shearing stress (Fig. 7.21c), as well as the values of \( \tau_{\text{max}} \) and of the corresponding normal stresses \( \sigma' \):

\[
\tau_{\text{max}} = \frac{P}{2A} \quad (7.18)
\]

In the case of torsion (Fig. 7.22a), we have \( \sigma_x = \sigma_y = 0 \) and \( \tau_{xy} = \tau_{\text{max}} = Tc/J. \) Points \( X \) and \( Y \), therefore, are located on the \( \tau \) axis, and Mohr’s circle is a circle of radius \( R = Tc/J \) centered at the origin (Fig. 7.22b). Points \( A \) and \( B \) define the principal planes (Fig. 7.22c) and the principal stresses:

\[
\sigma_{\text{max, min}} = \pm R = \pm \frac{Tc}{J} \quad (7.19)
\]
SAMPLE PROBLEM 7.2

For the state of plane stress shown, determine (a) the principal planes and the principal stresses, (b) the stress components exerted on the element obtained by rotating the given element counterclockwise through 30°.

SOLUTION

Construction of Mohr’s Circle. We note that on a face perpendicular to the x axis, the normal stress is tensile and the shearing stress tends to rotate the element clockwise; thus, we plot X at a point 100 units to the right of the vertical axis and 48 units above the horizontal axis. In a similar fashion, we examine the stress components on the upper face and plot point Y(60, –48). Joining points X and Y by a straight line, we define the center C of Mohr’s circle. The abscissa of C, which represents \( \sigma_{ave} \), and the radius R of the circle can be measured directly or calculated as follows:

\[
\sigma_{ave} = OC = \frac{1}{2}(\sigma_x + \sigma_y) = \frac{1}{2}(100 + 60) = 80 \text{ MPa}
\]

\[
R = \sqrt{(CF)^2 + (FX)^2} = \sqrt{(20)^2 + (48)^2} = 52 \text{ MPa}
\]

a. Principal Planes and Principal Stresses. We rotate the diameter XY clockwise through \( 2\theta_p \) until it coincides with the diameter AB. We have

\[
\tan 2\theta_p = \frac{XF}{CF} = \frac{48}{20} = 2.4 \quad 2\theta_p = 67.4^\circ \quad \theta_p = 33.7^\circ
\]

The principal stresses are represented by the abscissas of points A and B:

\[
\sigma_{max} = OA = OC + CA = 80 + 52 \quad \sigma_{max} = +132 \text{ MPa}
\]

\[
\sigma_{min} = OB = OC - BC = 80 - 52 \quad \sigma_{min} = +28 \text{ MPa}
\]

Since the rotation that brings XY into AB is clockwise, the rotation that brings Ox into the axis Ox corresponding to \( \sigma_{max} \) is also clockwise; we obtain the orientation shown for the principal planes.

b. Stress Components on Element Rotated 30°. Points X' and Y' on Mohr’s circle that correspond to the stress components on the rotated element are obtained by rotating XY counterclockwise through \( 2\theta = 60^\circ \). We find

\[
\phi = 180^\circ - 60^\circ = 67.4^\circ \quad \phi = 52.6^\circ
\]

\[
\sigma_x' = OK = OC - KC = 80 - 52 \cos 52.6^\circ \quad \sigma_x' = +48.4 \text{ MPa}
\]

\[
\sigma_y' = OL = OC + CL = 80 + 52 \cos 52.6^\circ \quad \sigma_y' = +111.6 \text{ MPa}
\]

\[
\tau_{x'y'} = KK' = 52 \sin 52.6^\circ \quad \tau_{x'y'} = 41.3 \text{ MPa}
\]

Since X’ is located above the horizontal axis, the shearing stress on the face perpendicular to Ox’ tends to rotate the element clockwise.
SAMPLE PROBLEM 7.3

A state of plane stress consists of a tensile stress \( \sigma_0 = 8 \text{ ksi} \) exerted on vertical surfaces and of unknown shearing stresses. Determine \((a)\) the magnitude of the shearing stress \( \tau_0 \) for which the largest normal stress is 10 ksi, \((b)\) the corresponding maximum shearing stress.

SOLUTION

Construction of Mohr’s Circle. We assume that the shearing stresses act in the senses shown. Thus, the shearing stress \( \tau_0 \) on a face perpendicular to the \( x \) axis tends to rotate the element clockwise and we plot the point \( X \) of coordinates 8 ksi and \( \tau_0 \) above the horizontal axis. Considering a horizontal face of the element, we observe that \( \sigma_y = 5 \text{ ksi} \) and that \( \tau_0 \) tends to rotate the element counterclockwise; thus, we plot point \( Y \) at a distance \( \tau_0 \) below \( O \).

We note that the abscissa of the center \( C \) of Mohr’s circle is \( \sigma_{\text{ave}} = \frac{1}{2}(\sigma_x + \sigma_y) = \frac{1}{2}(8 + 0) = 4 \text{ ksi} \)

The radius \( R \) of the circle is determined by observing that the maximum normal stress, \( \sigma_{\text{max}} = 10 \text{ ksi} \), is represented by the abscissa of point \( A \) and writing

\[
\sigma_{\text{max}} = \sigma_{\text{ave}} + R
\]

10ksi = 4ksi + R \quad \Rightarrow R = 6 ksi

\[\text{a. Shearing Stress } \tau_0.\] Considering the right triangle \( CFX \), we find

\[
\cos 2\theta_p = \frac{CF}{\sigma_y} = \frac{CF}{R} = \frac{4 \text{ ksi}}{6 \text{ ksi}} \quad 2\theta_p = 48.2^\circ \quad \Rightarrow \theta_p = 24.1^\circ
\]

\( \tau_0 = FX = R \sin 2\theta_p = (6 \text{ ksi}) \sin 48.2^\circ \quad \Rightarrow \tau_0 = 4.47 \text{ ksi} \)

\[\text{b. Maximum Shearing Stress.}\] The coordinates of point \( D \) of Mohr’s circle represent the maximum shearing stress and the corresponding normal stress.

\[
\tau_{\text{max}} = R = 6 \text{ ksi} \quad \sigma_{\text{max}} = 10 \text{ ksi}
\]

\[2\theta_p = 90^\circ - 2\theta_p = 90^\circ - 48.2^\circ = 41.8^\circ \quad \Rightarrow \theta_p = 20.9^\circ\]

The maximum shearing stress is exerted on an element that is oriented as shown in Fig. \( a \). (The element upon which the principal stresses are exerted is also shown.)

Note. If our original assumption regarding the sense of \( \tau_0 \) was reversed, we would obtain the same circle and the same answers, but the orientation of the elements would be as shown in Fig. \( b \).
7.31 Solve Probs. 7.5 and 7.9, using Mohr's circle.
7.32 Solve Probs. 7.7 and 7.11, using Mohr's circle.
7.33 Solve Prob. 7.10, using Mohr's circle.
7.34 Solve Prob. 7.12, using Mohr's circle.
7.35 Solve Prob. 7.13, using Mohr's circle.
7.36 Solve Prob. 7.14, using Mohr's circle.
7.37 Solve Prob. 7.15, using Mohr's circle.
7.38 Solve Prob. 7.16, using Mohr's circle.
7.39 Solve Prob. 7.17, using Mohr's circle.
7.40 Solve Prob. 7.18, using Mohr's circle.
7.41 Solve Prob. 7.19, using Mohr's circle.
7.42 Solve Prob. 7.20, using Mohr's circle.
7.43 Solve Prob. 7.21, using Mohr's circle.
7.44 Solve Prob. 7.22, using Mohr's circle.
7.45 Solve Prob. 7.23, using Mohr's circle.
7.46 Solve Prob. 7.24, using Mohr's circle.
7.47 Solve Prob. 7.25, using Mohr's circle.
7.48 Solve Prob. 7.26, using Mohr's circle.
7.49 Solve Prob. 7.27, using Mohr's circle.
7.50 Solve Prob. 7.28, using Mohr's circle.
7.51 Solve Prob. 7.29, using Mohr's circle.
7.52 Solve Prob. 7.30, using Mohr's circle.
7.53 Solve Prob. 7.30, using Mohr's circle and assuming that the weld forms an angle of 60° with the horizontal.
Determine the principal planes and the principal stresses for the state of plane stress resulting from the superposition of the two states of stress shown.

Fig. P7.54

Determine the principal planes and the principal stresses for the state of plane stress resulting from the superposition of the two states of stress shown.

Fig. P7.55

Determine the principal planes and the principal stresses for the state of plane stress resulting from the superposition of the two states of stress shown.

Fig. P7.56

Fig. P7.57
For the state of stress shown, determine the range of values of \( \theta \) for which the magnitude of the shearing stress \( \tau_{x'y'} \) is equal to or less than 8 ksi.

For the state of stress shown, determine the range of values of \( \theta \) for which the normal stress \( \sigma_{x'} \) is equal to or less than 50 MPa.

For the state of stress shown, determine the range of values of \( \theta \) for which the normal stress \( \sigma_{x'} \) is equal to or less than 100 MPa.

Fig. P7.58

Fig. P7.59 and P7.60

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For the state of stress shown, determine the range of values of \( \theta \) for which the normal stress \( \sigma_{x'} \) is equal to or less than 100 MPa.

For the element shown, determine the range of values of \( \tau_{xy} \) for which the maximum tensile stress is equal to or less than 60 MPa.

For the element shown, determine the range of values of \( \tau_{xy} \) for which the maximum in-plane shearing stress is equal to or less than 150 MPa.

For the state of stress shown it is known that the normal and shearing stresses are directed as shown and that \( \sigma_x = 14 \text{ ksi}, \sigma_y = 9 \text{ ksi}, \) and \( \sigma_{\text{min}} = 5 \text{ ksi}. \) Determine (a) the orientation of the principal planes, (b) the principal stress \( \sigma_{\text{max}} \), (c) the maximum in-plane shearing stress.

Fig. P7.61 and P7.62

Fig. P7.63
7.64 The Mohr’s circle shown corresponds to the state of stress given in Fig. 7.5a and b. Noting that \( \sigma_x = OC + (CX') \cos (2\theta_p - 2\theta) \) and that \( \tau_{x'y'} = (CX') \sin (2\theta_p - 2\theta) \), derive the expressions for \( \sigma_x \) and \( \tau_{x'y'} \) given in Eqs. (7.5) and (7.6), respectively. [Hint: Use \( \sin (A + B) = \sin A \cos B + \cos A \sin B \) and \( \cos (A + B) = \cos A \cos B - \sin A \sin B \).]

\[ \sigma_x = OC + (CX') \cos (2\theta_p - 2\theta) \]
\[ \tau_{x'y'} = (CX') \sin (2\theta_p - 2\theta) \]

7.65 (a) Prove that the expression \( \sigma_x \sigma_y = \tau_{x'y'}^2 \), where \( \sigma_x \), \( \sigma_y \), and \( \tau_{x'y'} \) are components of the stress along the rectangular axes \( x' \) and \( y' \), is independent of the orientation of these axes. Also, show that the given expression represents the square of the tangent drawn from the origin of the coordinates to Mohr’s circle. (b) Using the invariance property established in part a, express the shearing stress \( \tau_{xy} \) in terms of \( \sigma_x \), \( \sigma_y \), and the principal stresses \( \sigma_{\text{max}} \) and \( \sigma_{\text{min}} \).

7.5 GENERAL STATE OF STRESS

In the preceding sections, we have assumed a state of plane stress with \( \sigma_z = \tau_{zx} = \tau_{zy} = 0 \), and have considered only transformations of stress associated with a rotation about the \( z \) axis. We will now consider the general state of stress represented in Fig. 7.1a and the transformation of stress associated with the rotation of axes shown in Fig. 7.1b. However, our analysis will be limited to the determination of the normal stress \( \sigma_x \) on a plane of arbitrary orientation.

Consider the tetrahedron shown in Fig. 7.23. Three of its faces are parallel to the coordinate planes, while its fourth face, \( ABC \), is perpendicular to the line \( QN \). Denoting by \( \Delta A \) the area of face \( ABC \), and by \( \lambda_x \), \( \lambda_y \), \( \lambda_z \) the direction cosines of line \( QN \), we find that the areas of the faces perpendicular to the \( x, y \), and \( z \) axes are, respectively, \( (\Delta A)\lambda_x \), \( (\Delta A)\lambda_y \), and \( (\Delta A)\lambda_z \). If the state of stress at point \( Q \) is defined by the stress components \( \sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \) and \( \tau_{yz} \), then the forces exerted on the faces parallel to the coordinate planes can be obtained by multiplying the appropriate stress components by the area of each face (Fig. 7.24). On the other hand, the forces exerted on face \( ABC \) consist of a normal force of magnitude \( \sigma_x \Delta A \) directed along \( QN \), and of a shearing force of magnitude \( \tau \Delta A \) perpendicular to \( QN \) but of otherwise unknown direction. Note that, since \( QBC \),
QCA, and QAB, respectively, face the negative x, y, and z axes, the forces exerted on them must be shown with negative senses.

We now express that the sum of the components along QN of all the forces acting on the tetrahedron is zero. Observing that the component along QN of a force parallel to the x axis is obtained by multiplying the magnitude of that force by the direction cosine \( \lambda_x \) and that the components of forces parallel to the y and z axes are obtained in a similar way, we write

\[
\sum F_x = 0: \quad \sigma_x \Delta A - (\sigma_y \Delta A \lambda_y) \lambda_x - (\tau_{xy} \Delta A \lambda_x) \lambda_y - (\tau_{xz} \Delta A \lambda_x) \lambda_z
\]

\[-(\tau_{yx} \Delta A \lambda_y) \lambda_x - (\sigma_y \Delta A \lambda_y) \lambda_x - (\tau_{yz} \Delta A \lambda_y) \lambda_z - (\sigma_z \Delta A \lambda_z) \lambda_y - (\tau_{xz} \Delta A \lambda_z) \lambda_x = 0\]

Dividing through by \( \Delta A \) and solving for \( \sigma_n \), we have

\[
\sigma_n = \sigma_x \lambda_x^2 + \sigma_y \lambda_y^2 + \sigma_z \lambda_z^2 + 2\tau_{xy} \lambda_x \lambda_y + 2\tau_{yz} \lambda_y \lambda_z + 2\tau_{xz} \lambda_z \lambda_x \quad (7.20)
\]

We note that the expression obtained for the normal stress \( \sigma_n \) is a quadratic form in \( \lambda_x \), \( \lambda_y \), and \( \lambda_z \). It follows that we can select the coordinate axes in such a way that the right-hand member of Eq. (7.20) reduces to the three terms containing the squares of the direction cosines.† Denoting these axes by \( a \), \( b \), and \( c \), the corresponding normal stresses by \( \sigma_a \), \( \sigma_b \), and \( \sigma_c \), and the direction cosines of QN with respect to these axes by \( \lambda_a \), \( \lambda_b \), and \( \lambda_c \), we write

\[
\sigma_n = \sigma_a \lambda_a^2 + \sigma_b \lambda_b^2 + \sigma_c \lambda_c^2 \quad (7.21)
\]

The coordinate axes \( a \), \( b \), \( c \) are referred to as the principal axes of stress. Since their orientation is determined independently from the plane \( Q \), and thus upon the position of \( Q \), they have been represented in Fig. 7.25 as attached to \( Q \). The corresponding coordinate planes are known as the principal planes of stress, and the corresponding normal stresses \( \sigma_a \), \( \sigma_b \), and \( \sigma_c \), as the principal stresses at \( Q \).†

†In Sec. 9.16 of F. P. Beer and E. R. Johnston, Vector Mechanics for Engineers, 9th ed., McGraw-Hill Book Company, 2010, a similar quadratic form is found to represent the moment of inertia of a rigid body with respect to an arbitrary axis. It is shown in Sec. 9.17 that this form is associated with a quadratic surface, and that reducing the quadratic form to terms containing only the squares of the direction cosines is equivalent to determining the principal axes of that surface.

7.6 APPLICATION OF MOHR’S CIRCLE TO THE THREE-DIMENSIONAL ANALYSIS OF STRESS

If the element shown in Fig. 7.25 is rotated about one of the principal axes at \( Q \), say the \( c \) axis (Fig. 7.26), the corresponding transformation of stress can be analyzed by means of Mohr’s circle as if it were a transformation of plane stress. Indeed, the shearing stresses exerted on the faces perpendicular to the \( c \) axis remain equal to zero, and the normal stress \( \sigma_c \) is perpendicular to the plane \( ab \) in which the transformation takes place and, thus, does not affect this transformation. We therefore use the circle of diameter \( AB \) to determine the normal and shearing stresses exerted on the faces of the element as it is rotated about the \( c \) axis (Fig. 7.27). Similarly, circles of diameter \( BC \) and \( CA \) can be used to determine the stresses on the element as it is rotated about the \( a \) and \( b \) axes, respectively. While our analysis will be limited to rotations about the principal axes, it could be shown that any other transformation of axes would lead to stresses represented in Fig. 7.27 by a point located within the shaded area. Thus, the radius of the largest of the three circles yields the maximum value of the shearing stress at point \( Q \). Noting that the diameter of that circle is equal to the difference between \( \sigma_{\text{max}} \) and \( \sigma_{\text{min}} \), we write

\[
\tau_{\text{max}} = \frac{1}{2} \left[ \sigma_{\text{max}} - \sigma_{\text{min}} \right]
\]  

(7.22)

where \( \sigma_{\text{max}} \) and \( \sigma_{\text{min}} \) represent the algebraic values of the maximum and minimum stresses at point \( Q \).

Let us now return to the particular case of plane stress, which was discussed in Secs. 7.2 through 7.4. We recall that, if the \( x \) and \( y \) axes are selected in the plane of stress, we have \( \sigma_z = \tau_{zx} = \tau_{zy} = 0 \). This means that the \( z \) axis, i.e., the axis perpendicular to the plane of stress, is one of the three principal axes of stress. In a Mohr-circle diagram, this axis corresponds to the origin \( O \), where \( \sigma = \tau = 0 \). We also recall that the other two principal axes correspond to points \( A \) and \( B \) where Mohr’s circle for the \( xy \) plane intersects the \( \sigma \) axis. If \( A \) and \( B \) are located on opposite sides of the origin \( O \) (Fig. 7.28),...
the corresponding principal stresses represent the maximum and minimum normal stresses at point \( Q \), and the maximum shearing stress is equal to the maximum “in-plane” shearing stress. As noted in Sec. 7.3, the planes of maximum shearing stress correspond to points \( D \) and \( E \) of Mohr’s circle and are at 45° to the principal planes corresponding to points \( A \) and \( B \). They are, therefore, the shaded diagonal planes shown in Figs. 7.29a and b.

If, on the other hand, \( A \) and \( B \) are on the same side of \( O \), that is, if \( \sigma_a \) and \( \sigma_b \) have the same sign, then the circle defining \( \sigma_{\text{max}} \), \( \sigma_{\text{min}} \), and \( \tau_{\text{max}} \) is not the circle corresponding to a transformation of stress within the \( xy \) plane. If \( \sigma_a > \sigma_b > 0 \), as assumed in Fig. 7.30, we have \( \sigma_{\text{max}} = \sigma_a \), \( \sigma_{\text{min}} = 0 \), and \( \tau_{\text{max}} \) is equal to the radius of the circle defined by points \( O \) and \( A \), that is, \( \tau_{\text{max}} = \frac{1}{2} \sigma_{\text{max}} \). We also note that the normals \( Qd' \) and \( Qe' \) to the planes of maximum shearing stress are obtained by rotating the axis \( Qa \) through 45° within the \( za \) plane. Thus, the planes of maximum shearing stress are the shaded diagonal planes shown in Figs. 7.31a and b.
For the state of plane stress shown in Fig. 7.32, determine (a) the three principal planes and principal stresses, (b) the maximum shearing stress.

(a) Principal Planes and Principal Stresses. We construct Mohr’s circle for the transformation of stress in the \(xy\) plane (Fig. 7.33). Point \(X\) is plotted 6 units to the right of the \(\tau\) axis and 3 units above the \(\sigma\) axis (since the corresponding shearing stress tends to rotate the element clockwise). Point \(Y\) is plotted 3.5 units to the right of the \(\tau\) axis and 3 units below the \(\sigma\) axis. Drawing the line \(XY\), we obtain the center \(C\) of Mohr’s circle for the \(xy\) plane; its abscissa is

\[
\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = \frac{6 + 3.5}{2} = 4.75 \text{ ksi}
\]

Since the sides of the right triangle \(CFX\) are \(CF = 6 - 4.75 = 1.25 \text{ ksi}\) and \(FX = 3 \text{ ksi}\), the radius of the circle is

\[
R = CX = \sqrt{(1.25)^2 + (3)^2} = 3.25 \text{ ksi}
\]

The principal stresses in the plane of stress are

\[
\sigma_a = OA = OC + CA = 4.75 + 3.25 = 8.00 \text{ ksi}
\]
\[
\sigma_b = OB = OC - BC = 4.75 - 3.25 = 1.50 \text{ ksi}
\]

Since the faces of the element that are perpendicular to the \(z\) axis are free of stress, these faces define one of the principal planes, and the corresponding principal stress is \(\sigma_z = 0\). The other two principal planes are defined by points \(A\) and \(B\) on Mohr’s circle. The angle \(\theta_p\) through which the element should be rotated about the \(z\) axis to bring its faces to coincide with these planes (Fig. 7.34) is half the angle \(ACX\). We have

\[
\tan 2\theta_p = \frac{CF}{3} = \frac{1.25}{1.25} = 1
\]

\[
2\theta_p = 67.4^\circ \quad \theta_p = 33.7^\circ
\]

(b) Maximum Shearing Stress. We now draw the circles of diameter \(OB\) and \(OA\), which correspond respectively to rotations of the element about the \(a\) and \(b\) axes (Fig. 7.35). We note that the maximum shearing stress is equal to the radius of the circle of diameter \(OA\). We thus have

\[
\tau_{max} = \frac{1}{2} \sigma_a = \frac{1}{2}(8.00 \text{ ksi}) = 4.00 \text{ ksi}
\]

Since points \(D'\) and \(E'\), which define the planes of maximum shearing stress, are located at the ends of the vertical diameter of the circle corresponding to a rotation about the \(b\) axis, the faces of the element of Fig. 7.34 can be brought to coincide with the planes of maximum shearing stress through a rotation of \(45^\circ\) about the \(b\) axis.
**7.7 YIELD CRITERIA FOR DUCTILE MATERIALS UNDER PLANE STRESS**

Structural elements and machine components made of a ductile material are usually designed so that the material will not yield under the expected loading conditions. When the element or component is under uniaxial stress (Fig. 7.36), the value of the normal stress $\sigma_x$ that will cause the material to yield can be obtained readily from a tensile test conducted on a specimen of the same material, since the test specimen and the structural element or machine component are in the same state of stress. Thus, regardless of the actual mechanism that causes the material to yield, we can state that the element or component will be safe as long as $\sigma_x < \sigma_Y$, where $\sigma_Y$ is the yield strength of the test specimen.

On the other hand, when a structural element or machine component is in a state of plane stress (Fig. 7.37a), it is found convenient to use one of the methods developed earlier to determine the principal stresses $\sigma_a$ and $\sigma_b$ at any given point (Fig. 7.37b). The material can then be regarded as being in a state of biaxial stress at that point, since this state is different from the state of uniaxial stress found in a specimen subjected to a tensile test; it is clearly not possible to predict directly from such a test whether or not the structural element or machine component under investigation will fail. Some criterion regarding the actual mechanism of failure of the material must first be established, which will make it possible to compare the effects of both states of stress on the material. The purpose of this section is to present the two yield criteria most frequently used for ductile materials.

**Maximum-Shearing-Stress Criterion.** This criterion is based on the observation that yield in ductile materials is caused by slippage of the material along oblique surfaces and is due primarily to shearing stresses (cf. Sec. 2.3). According to this criterion, a given structural component is safe as long as the maximum value $\tau_{\text{max}}$ of the shearing stress in that component remains smaller than the corresponding value of the shearing stress in a tensile-test specimen of the same material as the specimen starts to yield.

Recalling from Sec. 1.11 that the maximum value of the shearing stress under a centric axial load is equal to half the value of the corresponding normal, axial stress, we conclude that the maximum shearing stress in a tensile-test specimen is $\frac{1}{2} \sigma_Y$ as the specimen starts to yield. On the other hand, we saw in Sec. 7.6 that, for plane stress, the maximum value $\tau_{\text{max}}$ of the shearing stress is equal to $\frac{1}{2} |\sigma_{\text{max}}|$ if the principal stresses are either both positive or both negative, and to $\frac{1}{2} |\sigma_{\text{max}} - \sigma_{\text{min}}|$ if the maximum stress is positive and the

![Fig. 7.36 Structural element under uniaxial stress.](image1)

![Fig. 7.37 Structural element in state of plane stress.](image2)
minimum stress negative. Thus, if the principal stresses $\sigma_a$ and $\sigma_b$ have the same sign, the maximum-shearing-stress criterion gives

$$|\sigma_a| < \sigma_Y \quad |\sigma_b| < \sigma_Y$$  \hspace{1cm} (7.23)

If the principal stresses $\sigma_a$ and $\sigma_b$ have opposite signs, the maximum-shearing-stress criterion yields

$$|\sigma_a - \sigma_b| < \sigma_Y$$  \hspace{1cm} (7.24)

The relations obtained have been represented graphically in Fig. 7.38. Any given state of stress will be represented in that figure by a point of coordinates $\sigma_a$ and $\sigma_b$, where $\sigma_a$ and $\sigma_b$ are the two principal stresses. If this point falls within the area shown in the figure, the structural component is safe. If it falls outside this area, the component will fail as a result of yield in the material. The hexagon associated with the initiation of yield in the material is known as Tresca’s hexagon after the French engineer Henri Edouard Tresca (1814–1885).

**Maximum-Distortion-Energy Criterion.** This criterion is based on the determination of the distortion energy in a given material, i.e., of the energy associated with changes in shape in that material (as opposed to the energy associated with changes in volume in the same material). According to this criterion, also known as the von Mises criterion, after the German-American applied mathematician Richard von Mises (1883–1953), a given structural component is safe as long as the maximum value of the distortion energy per unit volume in that material remains smaller than the distortion energy per unit volume in a tensile-test specimen of the same material. As you will see in Sec. 11.6, the distortion energy per unit volume in an isotropic material under plane stress is

$$u_d = \frac{1}{6G}(\sigma_a^2 - \sigma_a \sigma_b + \sigma_b^2)$$  \hspace{1cm} (7.25)

where $\sigma_a$ and $\sigma_b$ are the principal stresses and $G$ the modulus of rigidity. In the particular case of a tensile-test specimen that is starting to yield, we have $\sigma_a = \sigma_Y$, $\sigma_b = 0$, and $(u_d)_Y = \sigma_Y^2 / 6G$. Thus, the maximum-distortion-energy criterion indicates that the structural component is safe as long as $u_d < (u_d)_Y$, or

$$\sigma_a^2 - \sigma_a \sigma_b + \sigma_b^2 < \sigma_Y^2$$  \hspace{1cm} (7.26)

i.e., as long as the point of coordinates $\sigma_a$ and $\sigma_b$ falls within the area shown in Fig. 7.39. This area is bounded by the ellipse of equation

$$\sigma_a^2 - \sigma_a \sigma_b + \sigma_b^2 = \sigma_Y^2$$  \hspace{1cm} (7.27)

which intersects the coordinate axes at $\sigma_a = \pm \sigma_Y$ and $\sigma_b = \pm \sigma_Y$. We can verify that the major axis of the ellipse bisects the first and third quadrants and extends from $A (\sigma_a = \sigma_b = \sigma_Y)$ to $B (\sigma_a = \sigma_b = -\sigma_Y)$, while its minor axis extends from $C (\sigma_a = -\sigma_b = -0.577\sigma_Y)$ to $D (\sigma_a = -\sigma_b = 0.577\sigma_Y)$.

The maximum-shearing-stress criterion and the maximum-distortion-energy criterion are compared in Fig. 7.40. We note that the ellipse passes through the vertices of the hexagon. Thus, for the states of stress represented by these six points, the two criteria give...
the same results. For any other state of stress, the maximum-shearing-stress criterion is more conservative than the maximum-distortion-energy criterion, since the hexagon is located within the ellipse.

A state of particular interest is that associated with yield in a torsion test. We recall from Fig. 7.22 of Sec. 7.4 that, for torsion, \( \sigma_{\text{min}} = -\sigma_{\text{max}} \); thus, the corresponding points in Fig. 7.40 are located on the bisector of the second and fourth quadrants. It follows that yield occurs in a torsion test when \( s_a = 2 \frac{1}{\sqrt{2}} s_b = 0.577 s_Y \) according to the maximum-shearing-stress criterion, and when \( s_a = -s_b = \frac{1}{\sqrt{2}} s_Y \) according to the maximum-distortion-energy criterion. But, recalling again Fig. 7.22, we note that \( s_a \) and \( s_b \) must be equal in magnitude to \( \tau_{\text{max}} \), that is, to the value obtained from a torsion test for the yield strength \( \tau_Y \) of the material. Since the values of the yield strength \( \tau_Y \) in tension and of the yield strength \( \tau_Y \) in shear are given for various ductile materials in Appendix B, we can compute the ratio \( \tau_Y / \sigma_Y \) for these materials and verify that the values obtained range from 0.55 to 0.60. Thus, the maximum-distortion-energy criterion appears somewhat more accurate than the maximum-shearing-stress criterion as far as predicting yield in torsion is concerned.

**7.8 Fracture Criteria for Brittle Materials Under Plane Stress**

As we saw in Chap. 2, brittle materials are characterized by the fact that, when subjected to a tensile test, they fail suddenly through rupture—or fracture—without any prior yielding. When a structural element or machine component made of a brittle material is under uniaxial tensile stress, the value of the normal stress that causes it to fail is equal to the ultimate strength \( \sigma_U \) of the material as determined from a tensile test, since both the tensile-test specimen and the element or component under investigation are in the same state of stress. However, when a structural element or machine component is in a state of plane stress, it is found convenient to first determine the principal stresses \( \sigma_a \) and \( \sigma_b \) at any given point, and to use one of the criteria indicated in this section to predict whether or not the structural element or machine component will fail.

**Maximum-Normal-Stress Criterion.** According to this criterion, a given structural component fails when the maximum normal stress in that component reaches the ultimate strength \( \sigma_U \) obtained from the tensile test of a specimen of the same material. Thus, the structural component will be safe as long as the absolute values of the principal stresses \( \sigma_a \) and \( \sigma_b \) are both less than \( \sigma_U \):

\[
|\sigma_a| < \sigma_U \quad |\sigma_b| < \sigma_U
\]  

(7.28)

The maximum-normal-stress criterion can be expressed graphically as shown in Fig. 7.41. If the point obtained by plotting the values \( \sigma_a \) and \( \sigma_b \) of the principal stresses falls within the square area shown in the figure, the structural component is safe. If it falls outside that area, the component will fail.

The maximum-normal-stress criterion, also known as Coulomb's criterion, after the French physicist Charles Augustin de Coulomb.
(1736–1806), suffers from an important shortcoming, since it is based on the assumption that the ultimate strength of the material is the same in tension and in compression. As we noted in Sec. 2.3, this is seldom the case, because of the presence of flaws in the material, such as microscopic cracks or cavities, which tend to weaken the material in tension, while not appreciably affecting its resistance to compressive failure. Besides, this criterion makes no allowance for effects other than those of the normal stresses on the failure mechanism of the material.†

**Mohr’s Criterion.** This criterion, suggested by the German engineer Otto Mohr, can be used to predict the effect of a given state of plane stress on a brittle material, when results of various types of tests are available for that material.

Let us first assume that a tensile test and a compressive test have been conducted on a given material, and that the values \( \sigma_{UT} \) and \( \sigma_{UC} \) of the ultimate strength in tension and in compression have been determined for that material. The state of stress corresponding to the rupture of the tensile-test specimen can be represented on a Mohr-circle diagram by the circle intersecting the horizontal axis at \( O \) and \( \sigma_{UT} \) (Fig. 7.43a). Similarly, the state of stress corresponding to the failure of the compressive-test specimen can be represented by the circle intersecting the horizontal axis at \( O \) and \( \sigma_{UC} \). Clearly, a state of stress represented by a circle entirely contained in either of these circles will be safe. Thus, if both principal stresses are positive, the state of stress is safe as long as \( \sigma_a < \sigma_{UT} \) and \( \sigma_b < \sigma_{UT} \); if both principal stresses are negative, the state of stress is safe as long as \( |\sigma_a| < |\sigma_{UC}| \) and \( |\sigma_b| < |\sigma_{UC}| \). Plotting the point of coordinates \( \sigma_a \) and \( \sigma_b \) (Fig. 7.43b), we verify that the state of stress is safe as long as that point falls within one of the square areas shown in that figure.

In order to analyze the cases when \( \sigma_a \) and \( \sigma_b \) have opposite signs, we now assume that a torsion test has been conducted on the material and that its ultimate strength in shear, \( \tau_{UT} \), has been determined. Drawing the circle centered at \( O \) representing the state of stress corresponding to the failure of the torsion-test specimen (Fig. 7.44a), we observe that any state of stress represented by a circle entirely contained in that circle is also safe. Mohr’s criterion is a logical extension of this observation: According to Mohr’s criterion, a state of stress is safe if it is represented by a circle located entirely within the area bounded

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†Another failure criterion known as the maxmum-normal-strain criterion, or Saint-Venant’s criterion, was widely used during the nineteenth century. According to this criterion, a given structural component is safe as long as the maximum value of the normal strain in that component remains smaller than the value \( \varepsilon_U \) of the strain at which a tensile-test specimen of the same material will fail. But, as will be shown in Sec. 7.12, the strain is maximum along one of the principal axes of stress, if the deformation is elastic and the material homogeneous and isotropic. Thus, denoting by \( \varepsilon_a \) and \( \varepsilon_b \) the values of the normal strain along the principal axes in the plane of stress, we write

\[
|\varepsilon_a| < \varepsilon_U \quad |\varepsilon_b| < \varepsilon_U
\]

(7.29)

Making use of the generalized Hooke’s law (Sec. 2.12), we could express these relations in terms of the principal stresses \( \sigma_a \) and \( \sigma_b \) and the ultimate strength \( \sigma_U \) of the material. We would find that, according to the maximum-normal-strain criterion, the structural component is safe as long as the point obtained by plotting \( \sigma_a \) and \( \sigma_b \) falls within the area shown in Fig. 7.42 where \( \nu \) is Poisson’s ratio for the given material.
by the envelope of the circles corresponding to the available data. The remaining portions of the principal-stress diagram can now be obtained by drawing various circles tangent to this envelope, determining the corresponding values of \( \sigma_a \) and \( \sigma_b \), and plotting the points of coordinates \( \sigma_a \) and \( \sigma_b \) (Fig. 7.44b).

More accurate diagrams can be drawn when additional test results, corresponding to various states of stress, are available. If, on the other hand, the only available data consists of the ultimate strengths \( \sigma_{UT} \) and \( \sigma_{UC} \), the envelope in Fig. 7.44a is replaced by the tangents \( AB \) and \( A'B' \) to the circles corresponding respectively to failure in tension and failure in compression (Fig. 7.45a). From the similar triangles drawn in that figure, it may be shown that the radius \( R \) of the center \( C \) of a circle tangent to \( AB \) and \( A'B' \) is linearly related to its radius \( R' \). Since \( \sigma_a = OC + R \) and \( \sigma_b = OC - R \), it follows that \( \sigma_a \) and \( \sigma_b \) are also linearly related. Thus, the shaded area corresponding to this simplified Mohr's criterion is bounded by straight lines in the second and fourth quadrants (Fig. 7.45b).

Note that in order to determine whether a structural component will be safe under a given loading, the state of stress should be calculated at all critical points of the component, i.e., at all points where stress concentrations are likely to occur. This can be done in a number of cases by using the stress-concentration factors given in Figs. 2.60, 3.29, 4.27, and 4.28. There are many instances, however, when the theory of elasticity must be used to determine the state of stress at a critical point.

Special care should be taken when macroscopic cracks have been detected in a structural component. While it can be assumed that the test specimen used to determine the ultimate tensile strength of the material contained the same type of flaws (i.e., microscopic cracks or cavities) as the structural component under investigation, the specimen was certainly free of any detectable macroscopic cracks. When a crack is detected in a structural component, it is necessary to determine whether that crack will tend to propagate under the expected loading condition and cause the component to fail, or whether it will remain stable. This requires an analysis involving the energy associated with the growth of the crack. Such an analysis is beyond the scope of this text and should be carried out by the methods of fracture mechanics.

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**Figure 7.44** Mohr's criterion.

**Figure 7.45** Simplified Mohr's criterion.
SAMPLE PROBLEM 7.4

The state of plane stress shown occurs at a critical point of a steel machine component. As a result of several tensile tests, it has been found that the tensile yield strength is $\sigma_Y = 250$ MPa for the grade of steel used. Determine the factor of safety with respect to yield, using (a) the maximum-shearing-stress criterion, and (b) the maximum-distortion-energy criterion.

SOLUTION

Mohr’s Circle. We construct Mohr’s circle for the given state of stress and find

$$\sigma_{sys} = OC = \frac{1}{2} (\sigma_x + \sigma_y) = \frac{1}{2} (80 - 40) = 20 \text{ MPa}$$

$$\tau_m = R = \sqrt{(CF)^2 + (FX)^2} = \sqrt{(60)^2 + (25)^2} = 65 \text{ MPa}$$

Principal Stresses

$$\sigma_a = OC + CA = 20 + 65 = +85 \text{ MPa}$$

$$\sigma_b = OC - BC = 20 - 65 = -45 \text{ MPa}$$

a. Maximum-Shearing-Stress Criterion. Since for the grade of steel used the tensile strength is $\sigma_Y = 250$ MPa, the corresponding shearing stress at yield is

$$\tau_y = \frac{1}{2} \sigma_Y = \frac{1}{2} (250 \text{ MPa}) = 125 \text{ MPa}$$

For $\tau_m = 65$ MPa:

$$F.S. = \frac{\tau_y}{\tau_m} = \frac{125 \text{ MPa}}{65 \text{ MPa}} = 1.92 \quad \text{F.S.} = 1.92$$

b. Maximum-Distortion-Energy Criterion. Introducing a factor of safety into Eq. (7.26), we write

$$\sigma_a^2 - \sigma_a \sigma_b + \sigma_b^2 = \left(\frac{\sigma_Y}{F.S.}\right)^2$$

For $\sigma_a = +85$ MPa, $\sigma_b = -45$ MPa, and $\sigma_Y = 250$ MPa, we have

$$250^2 - (85)(-45) + (45)^2 = \left(\frac{250}{F.S.}\right)^2$$

$$114.3 = \frac{250}{F.S.} \quad F.S. = 2.19$$

Comment. For a ductile material with $\sigma_Y = 250$ MPa, we have drawn the hexagon associated with the maximum-shearing-stress criterion and the ellipse associated with the maximum-distortion-energy criterion. The given state of plane stress is represented by point H of coordinates $\sigma_a = 85$ MPa and $\sigma_b = -45$ MPa. We note that the straight line drawn through points O and H intersects the hexagon at point T and the ellipse at point M. For each criterion, the value obtained for F.S. can be verified by measuring the line segments indicated and computing their ratios:

(a) $F.S. = \frac{OT}{OH} = 1.92$  (b) $F.S. = \frac{OM}{OH} = 2.19$
PROBLEMS

7.66 For the state of plane stress shown, determine the maximum shearing stress when (a) \( \sigma_x = 6 \text{ ksi} \) and \( \sigma_y = 18 \text{ ksi} \), (b) \( \sigma_x = 14 \text{ ksi} \) and \( \sigma_y = 2 \text{ ksi} \). (Hint: Consider both in-plane and out-of-plane shearing stresses.)

![Diagram of a cube with stresses](Fig. P7.66 and P7.67)

7.67 For the state of plane stress shown, determine the maximum shearing stress when (a) \( \sigma_x = 0 \) and \( \sigma_y = 12 \text{ ksi} \), (b) \( \sigma_x = 21 \text{ ksi} \) and \( \sigma_y = 9 \text{ ksi} \). (Hint: Consider both in-plane and out-of-plane shearing stresses.)

![Diagram of a cube with stresses](Fig. P7.68 and P7.69)

7.68 For the state of stress shown, determine the maximum shearing stress when (a) \( \sigma_y = 40 \text{ MPa} \), (b) \( \sigma_y = 120 \text{ MPa} \). (Hint: Consider both in-plane and out-of-plane shearing stresses.)

7.69 For the state of stress shown, determine the maximum shearing stress when (a) \( \sigma_y = 20 \text{ MPa} \), (b) \( \sigma_y = 140 \text{ MPa} \). (Hint: Consider both in-plane and out-of-plane shearing stresses.)

7.70 and 7.71 For the state of stress shown, determine the maximum shearing stress when (a) \( \sigma_z = +4 \text{ ksi} \), (b) \( \sigma_z = -4 \text{ ksi} \), (c) \( \sigma_z = 0 \).

![Diagram of a cube with stresses](Fig. P7.70 and Fig. P7.71)
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**7.72 and 7.73** For the state of stress shown, determine the maximum shearing stress when (a) $\sigma_z = 0$, (b) $\sigma_z = +45$ MPa, (c) $\sigma_z = -45$ MPa.

![Fig. P7.72](image)

![Fig. P7.73](image)

**7.74** For the state of stress shown, determine two values of $\sigma_y$ for which the maximum shearing stress is 10 ksi.

![Fig. P7.74](image)

![Fig. P7.75](image)

**7.75** For the state of stress shown, determine two values of $\sigma_y$ for which the maximum shearing stress is 73 MPa.

**7.76** For the state of stress shown, determine the value of $\tau_{yz}$ for which the maximum shearing stress is (a) 10 ksi, (b) 8.25 ksi.

![Fig. P7.76](image)
7.77 For the state of stress shown, determine the value of \( \tau_{xy} \) for which the maximum shearing stress is (a) 60 MPa, (b) 78 MPa.

![Fig. P7.77](image1)

7.78 For the state of stress shown, determine two values of \( \sigma_y \) for which the maximum shearing stress is 80 MPa.

7.79 For the state of stress shown, determine the range of values of \( \tau_{xz} \) for which the maximum shearing stress is equal to or less than 60 MPa.

7.80 For the state of stress of Prob. 7.69, determine (a) the value of \( \sigma_y \) for which the maximum shearing stress is as small as possible, (b) the corresponding value of the shearing stress.

7.81 The state of plane stress shown occurs in a machine component made of a steel with \( \sigma_Y = 325 \) MPa. Using the maximum-distortion-energy criterion, determine whether yield will occur when (a) \( \sigma_0 = 200 \) MPa, (b) \( \sigma_0 = 240 \) MPa, (c) \( \sigma_0 = 280 \) MPa. If yield does not occur, determine the corresponding factor of safety.

7.82 Solve Prob. 7.81, using the maximum-shearing-stress criterion.

7.83 The state of plane stress shown occurs in a machine component made of a steel with \( \sigma_Y = 45 \) ksi. Using the maximum-distortion-energy criterion, determine whether yield will occur when (a) \( \tau_{xy} = 9 \) ksi, (b) \( \tau_{xy} = 18 \) ksi, (c) \( \tau_{xy} = 20 \) ksi. If yield does not occur, determine the corresponding factor of safety.

7.84 Solve Prob. 7.83, using the maximum-shearing-stress criterion.
7.85 The 36-mm-diameter shaft is made of a grade of steel with a 250-MPa tensile yield stress. Using the maximum-shearing-stress criterion, determine the magnitude of the torque $T$ for which yield occurs when $P = 200$ kN.

7.86 Solve Prob. 7.85, using the maximum-distortion-energy criterion.

7.87 The 1.75-in.-diameter shaft $AB$ is made of a grade of steel for which the yield strength is $\sigma_Y = 36$ ksi. Using the maximum-shearing-stress criterion, determine the magnitude of the force $P$ for which yield occurs when $T = 15$ kip $\cdot$ in.

7.88 Solve Prob. 7.87, using the maximum-distortion-energy criterion.

7.89 and 7.90 The state of plane stress shown is expected to occur in an aluminum casting. Knowing that for the aluminum alloy used in 7.89 and 7.90, $\sigma_{UT} = 80$ MPa and $\sigma_{UC} = 300$ MPa and using Mohr’s criterion, determine whether rupture of the casting will occur.

7.91 and 7.92 The state of plane stress shown is expected to occur in an aluminum casting. Knowing that for the aluminum alloy used in 7.91 and 7.92, $\sigma_{UT} = 10$ ksi and $\sigma_{UC} = 30$ ksi and using Mohr’s criterion, determine whether rupture of the casting will occur.
7.93 The state of plane stress shown will occur at a critical point in an aluminum casting that is made of an alloy for which $\sigma_{UT} = 10$ ksi and $\sigma_{UC} = 25$ ksi. Using Mohr’s criterion, determine the shearing stress $\tau_0$ for which failure should be expected.

![Fig. P7.93](image)

7.94 The state of plane stress shown will occur at a critical point in a pipe made of an aluminum alloy for which $\sigma_{UT} = 75$ MPa and $\sigma_{UC} = 150$ MPa. Using Mohr’s criterion, determine the shearing stress $\tau_0$ for which failure should be expected.

![Fig. P7.94](image)

7.95 The cast-aluminum rod shown is made of an alloy for which $\sigma_{UT} = 60$ MPa and $\sigma_{UC} = 120$ MPa. Using Mohr’s criterion, determine the magnitude of the torque $T$ for which failure should be expected.

![Fig. P7.95](image)

7.96 The cast-aluminum rod shown is made of an alloy for which $\sigma_{UT} = 70$ MPa and $\sigma_{UC} = 175$ MPa. Knowing that the magnitude $T$ of the applied torques is slowly increased and using Mohr’s criterion, determine the shearing stress $\tau_0$ that should be expected at rupture.

![Fig. P7.96](image)

7.97 A machine component is made of a grade of cast iron for which $\sigma_{UT} = 8$ ksi and $\sigma_{UC} = 20$ ksi. For each of the states of stress shown, and using Mohr’s criterion, determine the normal stress $\sigma_0$ at which rupture of the component should be expected.

![Fig. P7.97](image)
7.9 STRESSES IN THIN-WALLED PRESSURE VESSELS

Thin-walled pressure vessels provide an important application of the analysis of plane stress. Since their walls offer little resistance to bending, it can be assumed that the internal forces exerted on a given portion of wall are tangent to the surface of the vessel (Fig. 7.46). The resulting stresses on an element of wall will thus be contained in a plane tangent to the surface of the vessel.

Our analysis of stresses in thin-walled pressure vessels will be limited to the two types of vessels most frequently encountered: cylindrical pressure vessels and spherical pressure vessels (Photos 7.3 and 7.4).

Consider a cylindrical vessel of inner radius $r$ and wall thickness $t$ containing a fluid under pressure (Fig. 7.47). We propose to determine the stresses exerted on a small element of wall with sides respectively parallel and perpendicular to the axis of the cylinder. Because of the axisymmetry of the vessel and its contents, it is clear that no shearing stress is exerted on the element. The normal stresses $\sigma_1$ and $\sigma_2$ shown in Fig. 7.47 are therefore principal stresses. The stress $\sigma_1$ is known as the hoop stress, because it is the type of stress found in hoops used to hold together the various slats of a wooden barrel, and the stress $\sigma_2$ is called the longitudinal stress.

In order to determine the hoop stress $\sigma_1$, we detach a portion of the vessel and its contents bounded by the $xy$ plane and by two planes parallel to the $yz$ plane at a distance $\Delta \xi$ from each other (Fig. 7.48). The forces parallel to the $z$ axis acting on the free body defined in this fashion consist of the elementary internal forces $\sigma_1 \, dA$ on the wall sections, and of the elementary pressure forces $p \, dA$ exerted on the portion of fluid included in the free body. Note that $p$ denotes the gage pressure of the fluid, i.e., the excess of the inside pressure over the outside atmospheric pressure. The resultant of the internal forces $\sigma_1 \, dA$ is equal to the product of $\sigma_1$ and of the cross-sectional area $2\pi \, \Delta \xi$ of the wall, while the resultant of the pressure forces $p \, dA$ is equal to the product of $p$ and of the area $2\pi \, \Delta \xi$. Writing the equilibrium equation $\Sigma F_z = 0$, we have
and, solving for the hoop stress $\sigma_1$,

$$\sigma_1 = \frac{pr}{t} \quad (7.30)$$

To determine the longitudinal stress $\sigma_2$, we now pass a section perpendicular to the $x$ axis and consider the free body consisting of the portion of the vessel and its contents located to the left of the section.

![Free body diagram](http://www.opoosoft.com)

(Fig. 7.49). The forces acting on this free body are the elementary internal forces $\sigma_2 \, dA$ on the wall section and the elementary pressure forces $p \, dA$ exerted on the portion of the vessel and its contents located to the left of the section. Noting that the area of the fluid section is $\pi r^2$ and that the area of the wall section can be obtained by multiplying the circumference $2\pi r$ of the cylinder by its wall thickness $t$, we write the equilibrium equation:

$$\Sigma F_z = 0: \quad \sigma_2 (2\pi r \, \Delta z) - p(2r \, \Delta x) = 0$$

and, solving for the longitudinal stress $\sigma_2$,

$$\sigma_2 = \frac{pr}{2t} \quad (7.31)$$

We note from Eqs. (7.30) and (7.31) that the hoop stress $\sigma_1$ is twice as large as the longitudinal stress $\sigma_2$:

$$\sigma_1 = 2\sigma_2 \quad (7.32)$$

†Using the mean radius of the wall section, $r_m = r + \frac{1}{2} t$, in computing the resultant of the forces on that section, we would obtain a more accurate value of the longitudinal stress, namely,

$$\sigma_2 = \frac{pr}{2} \frac{1}{1 + \frac{t}{2r}} \quad (7.31')$$

However, for a thin-walled pressure vessel, the term $t/2r$ is sufficiently small to allow the use of Eq. (7.31) for engineering design and analysis. If a pressure vessel is not thin-walled (i.e., if $t/2r$ is not small), the stresses $\sigma_1$ and $\sigma_2$ vary across the wall and must be determined by the methods of the theory of elasticity.
Transformations of Stress and Strain

Drawing Mohr’s circle through the points A and B that correspond respectively to the principal stresses $\sigma_1$ and $\sigma_2$ (Fig. 7.50), and recalling that the maximum in-plane shearing stress is equal to the radius of this circle, we have

$$\tau_{\text{max (in plane)}} = \frac{1}{2} \sigma_2 = \frac{pr}{4t} \quad (7.33)$$

This stress corresponds to points D and E and is exerted on an element obtained by rotating the original element of Fig. 7.47 through $45^\circ$ within the plane tangent to the surface of the vessel. The maximum shearing stress in the wall of the vessel, however, is larger. It is equal to the radius of the circle of diameter OA and corresponds to a rotation of $45^\circ$ about a longitudinal axis and out of the plane of stress.$^\dagger$ We have

$$\tau_{\text{max}} = \sigma_2 = \frac{pr}{2t} \quad (7.34)$$

We now consider a spherical vessel of inner radius r and wall thickness t, containing a fluid under a gage pressure p. For reasons of symmetry, the stresses exerted on the four faces of a small element of wall must be equal (Fig. 7.51). We have

$$\sigma_1 = \sigma_2 \quad (7.35)$$

To determine the value of the stress, we pass a section through the center C of the vessel and consider the free body consisting of the portion of the vessel and its contents located to the left of the section (Fig. 7.52). The equation of equilibrium for this free body is the same as for the free body of Fig. 7.49. We thus conclude that, for a spherical vessel,

$$\sigma_1 = \sigma_2 = \frac{pr}{2t} \quad (7.36)$$

Since the principal stresses $\sigma_1$ and $\sigma_2$ are equal, Mohr’s circle for transformations of stress within the plane tangent to the surface of the vessel reduces to a point (Fig. 7.53); we conclude that the in-plane normal stress is constant and that the in-plane maximum shearing stress is zero. The maximum shearing stress in the wall of the vessel, however, is not zero; it is equal to the radius of the circle of diameter OA and corresponds to a rotation of $45^\circ$ out of the plane of stress. We have

$$\tau_{\text{max}} = \frac{1}{2} \sigma_1 = \frac{pr}{4t} \quad (7.37)$$

$^\dagger$It should be observed that, while the third principal stress is zero on the outer surface of the vessel, it is equal to $\sigma_1$ on the inner surface, and is represented by a point $C(-p, 0)$ on a Mohr-circle diagram. Thus, close to the inside surface of the vessel, the maximum shearing stress is equal to the radius of a circle of diameter CA, and we have

$$\tau_{\text{max}} = \frac{1}{2} (\sigma_1 + p) = \frac{pr}{2t} \left( 1 + \frac{t}{r} \right)$$

For a thin-walled vessel, however, the term $t/r$ is small, and we can neglect the variation of $\tau_{\text{max}}$ across the wall section. This remark also applies to spherical pressure vessels.
SAMPLE PROBLEM 7.5

A compressed-air tank is supported by two cradles as shown; one of the cradles is designed so that it does not exert any longitudinal force on the tank. The cylindrical body of the tank has a 30-in. outer diameter and is fabricated from a \( \frac{3}{8} \)-in. steel plate by butt welding along a helix that forms an angle of 25° with a transverse plane. The end caps are spherical and have a uniform wall thickness of \( \frac{5}{16} \) in. For an internal gage pressure of 180 psi, determine (a) the normal stress and the maximum shearing stress in the spherical caps. (b) the stresses in directions perpendicular and parallel to the helical weld.

SOLUTION

a. Spherical Cap. Using Eq. (7.36), we write

\[
p = 180 \text{ psi}, \quad t = \frac{3}{8} \text{ in.} = 0.3125 \text{ in.}, \quad r = 15 - 0.3125 = 14.688 \text{ in.}
\]

\[
\sigma_1 = \sigma_2 = \frac{pr}{2t} = \frac{(180 \text{ psi})(14.688 \text{ in.})}{2(0.3125 \text{ in.})} \quad \sigma = 4230 \text{ psi}
\]

We note that for stresses in a plane tangent to the cap, Mohr’s circle reduces to a point (A, B) on the horizontal axis and that all in-plane shearing stresses are zero. On the surface of the cap the third principal stress is zero and corresponds to point O. On a Mohr’s circle of diameter AO, point D represents the maximum shearing stress; it occurs on planes at 45° to the plane tangent to the cap.

\[
\tau_{\text{max}} = \frac{1}{2} (4230 \text{ psi}) \quad \tau_{\text{max}} = 2115 \text{ psi}
\]

b. Cylindrical Body of the Tank. We first determine the hoop stress \( \sigma_1 \) and the longitudinal stress \( \sigma_2 \). Using Eqs. (7.30) and (7.32), we write

\[
p = 180 \text{ psi}, \quad t = \frac{3}{8} \text{ in.} = 0.375 \text{ in.}, \quad r = 15 - 0.375 = 14.625 \text{ in.}
\]

\[
\sigma_1 = \frac{pr}{t} = \frac{(180 \text{ psi})(14.625 \text{ in.})}{0.375 \text{ in.}} = 7020 \text{ psi} \quad \sigma_2 = \frac{1}{2} \sigma_1 = 3510 \text{ psi}
\]

\[
\sigma_{\text{ave}} = \frac{1}{2} (\sigma_1 + \sigma_2) = 5265 \text{ psi} \quad R = \frac{1}{2} (\sigma_1 - \sigma_2) = 1755 \text{ psi}
\]

Stresses at the Weld. Noting that both the hoop stress and the longitudinal stress are principal stresses, we draw Mohr’s circle as shown.

An element having a face parallel to the weld is obtained by rotating the face perpendicular to the axis Ob counterclockwise through 25°. Therefore, on Mohr’s circle we locate the point X’ corresponding to the stress components on the weld by rotating radius CB counterclockwise through 2\( \theta \) = 50°.

\[
\sigma_w = \sigma_{\text{ave}} - R \cos 50^\circ = 5265 - 1755 \cos 50^\circ \quad \sigma_w = +4140 \text{ psi}
\]

\[
\tau_w = R \sin 50^\circ = 1755 \sin 50^\circ \quad \tau_w = 1344 \text{ psi}
\]

Since X’ is below the horizontal axis, \( \tau_w \) tends to rotate the element counterclockwise.
7.98 A spherical gas container made of steel has a 5-m outer diameter and a wall thickness of 6 mm. Knowing that the internal pressure is 350 kPa, determine the maximum normal stress and the maximum shearing stress in the container.

7.99 The maximum gage pressure is known to be 8 MPa in a spherical steel pressure vessel having a 250-mm outer diameter and a 6-mm wall thickness. Knowing that the ultimate stress in the steel used is \( \sigma_u = 400 \) MPa, determine the factor of safety with respect to tensile failure.

7.100 A basketball has a 9.5-in. outer diameter and a 0.125-in. wall thickness. Determine the normal stress in the wall when the basketball is inflated to a 9-psi gage pressure.

7.101 A spherical pressure vessel of 900-mm outer diameter is to be fabricated from a steel having an ultimate stress \( \sigma_u = 400 \) MPa. Knowing that a factor of safety of 4.0 is desired and that the gage pressure can reach 3.5 MPa, determine the smallest wall thickness that should be used.

7.102 A spherical pressure vessel has an outer diameter of 10 ft and a wall thickness of 0.5 in. Knowing that for the steel used \( \sigma_{uI} = 12 \) ksi, \( E = 29 \times 10^6 \) psi, and \( v = 0.29 \), determine (a) the allowable gage pressure, (b) the corresponding increase in the diameter of the vessel.

7.103 A spherical gas container having an outer diameter of 5 m and a wall thickness of 22 mm is made of steel for which \( E = 200 \) GPa and \( v = 0.29 \). Knowing that the gage pressure in the container is increased from zero to 1.7 MPa, determine (a) the maximum normal stress in the container, (b) the corresponding increase in the diameter of the container.

7.104 A steel penstock has a 750-mm outer diameter, a 12-mm wall thickness, and connects a reservoir at A with a generating station at B. Knowing that the density of water is 1000 kg/m\(^3\), determine the maximum normal stress and the maximum shearing stress in the penstock under static conditions.

7.105 A steel penstock has a 750-mm outer diameter and connects a reservoir at A with a generating station at B. Knowing that the density of water is 1000 kg/m\(^3\) and that the allowable normal stress in the steel is 85 MPa, determine the smallest thickness that can be used for the penstock.

7.106 The bulk storage tank shown in Photo 7.3 has an outer diameter of 3.5 m and a wall thickness of 18 mm. At a time when the internal pressure of the tank is 1.5 MPa, determine the maximum normal stress and the maximum shearing stress in the tank.
7.107 Determine the largest internal pressure that can be applied to a cylindrical tank of 5.5-ft outer diameter and \( \frac{7}{8} \)-in. wall thickness if the ultimate normal stress of the steel used is 65 ksi and a factor of safety of 5.0 is desired.

7.108 A cylindrical storage tank contains liquefied propane under a pressure of 1.5 MPa at a temperature of 38°C. Knowing that the tank has an outer diameter of 320 mm and a wall thickness of 3 mm, determine the maximum normal stress and the maximum shearing stress in the tank.

7.109 The unpressurized cylindrical storage tank shown has a \( \frac{3}{16} \)-in. wall thickness and is made of steel having a 60-ksi ultimate strength in tension. Determine the maximum height \( h \) to which it can be filled with water if a factor of safety of 4.0 is desired. (Specific weight of water = 62.4 lb/ft\(^3\).)

7.110 For the storage tank of Prob. 7.109, determine the maximum normal stress and the maximum shearing stress in the cylindrical wall when the tank is filled to capacity (\( h = 48 \) ft).

7.111 A standard-weight steel pipe of 12-in. nominal diameter carries water under a pressure of 400 psi. (a) Knowing that the outside diameter is 12.75 in. and the wall thickness is 0.375 in., determine the maximum tensile stress in the pipe. (b) Solve part (a), assuming an extra-strong pipe is used, of 12.75-in. outside diameter and 0.5-in. wall thickness.

7.112 The pressure tank shown has a 8-mm wall thickness and butt-welded seams forming an angle \( \beta = 20^\circ \) with a transverse plane. For a gage pressure of 600 kPa, determine, (a) the normal stress perpendicular to the weld, (b) the shearing stress parallel to the weld.

7.113 For the tank of Prob. 7.112, determine the largest allowable gage pressure, knowing that the allowable normal stress perpendicular to the weld is 120 MPa and the allowable shearing stress parallel to the weld is 80 MPa.

7.114 For the tank of Prob. 7.112, determine the range of values of \( \beta \) that can be used if the shearing stress parallel to the weld is not to exceed 12 MPa when the gage pressure is 600 kPa.

7.115 The steel pressure tank shown has a 750-mm inner diameter and a 9-mm wall thickness. Knowing that the butt-welded seams form an angle \( \beta = 50^\circ \) with the longitudinal axis of the tank and that the gage pressure in the tank is 1.5 MPa, determine, (a) the normal stress perpendicular to the weld, (b) the shearing stress parallel to the weld.

7.116 The pressurized tank shown was fabricated by welding strips of plate along a helix forming an angle \( \beta \) with a transverse plane. Determine the largest value of \( \beta \) that can be used if the normal stress perpendicular to the weld is not to be larger than 85 percent of the maximum stress in the tank.
7.117 The cylindrical portion of the compressed-air tank shown is fabricated of 0.25-in.-thick plate welded along a helix forming an angle $\beta = 30^\circ$ with the horizontal. Knowing that the allowable stress normal to the weld is 10.5 ksi, determine the largest gage pressure that can be used in the tank.

7.118 For the compressed-air tank of Prob. 7.117, determine the gage pressure that will cause a shearing stress parallel to the weld of 4 ksi.

7.119 Square plates, each of 0.5-in. thickness, can be bent and welded together in either of the two ways shown to form the cylindrical portion of a compressed-air tank. Knowing that the allowable normal stress perpendicular to the weld is 12 ksi, determine the largest allowable gage pressure in each case.

7.120 The compressed-air tank $AB$ has an inner diameter of 450 mm and a uniform wall thickness of 6 mm. Knowing that the gage pressure inside the tank is 1.2 MPa, determine the maximum normal stress and the maximum in-plane shearing stress at point $a$ on the top of the tank.

7.121 For the compressed-air tank and loading of Prob. 7.120, determine the maximum normal stress and the maximum in-plane shearing stress at point $b$ on the top of the tank.
7.122 The compressed-air tank AB has a 250-mm outside diameter and an 8-mm wall thickness. It is fitted with a collar by which a 40-kN force P is applied at B in the horizontal direction. Knowing that the gage pressure inside the tank is 5 MPa, determine the maximum normal stress and the maximum shearing stress at point K.

7.123 In Prob. 7.122, determine the maximum normal stress and the maximum shearing stress at point L.

7.124 A pressure vessel of 10-in. inner diameter and 0.25-in. wall thickness is fabricated from a 4-ft section of spirally-welded pipe AB and is equipped with two rigid end plates. The gage pressure inside the vessel is 300 psi and 10-kip centric axial forces P and P' are applied to the end plates. Determine (a) the normal stress perpendicular to the weld, (b) the shearing stress parallel to the weld.

7.125 Solve Prob. 7.124, assuming that the magnitude P of the two forces is increased to 30 kips.

7.126 A brass ring of 5-in. outer diameter and 0.25-in. thickness fits exactly inside a steel ring of 5-in. inner diameter and 0.125-in. thickness when the temperature of both rings is 50°F. Knowing that the temperature of both rings is then raised to 125°F, determine (a) the tensile stress in the steel ring, (b) the corresponding pressure exerted by the brass ring on the steel ring.

7.127 Solve Prob. 7.126, assuming that the brass ring is 0.125 in. thick and the steel ring is 0.25 in. thick.
7.10 TRANSFORMATION OF PLANE STRAIN

Transformations of strain under a rotation of the coordinate axes will now be considered. Our analysis will first be limited to states of plane strain, i.e., to situations where the deformations of the material take place within parallel planes, and are the same in each of these planes. If the z axis is chosen perpendicular to the planes in which the deformations take place, we have $\varepsilon_z = \gamma_{zx} = \gamma_{zy} = 0$, and the only remaining strain components are $\varepsilon_x$, $\varepsilon_y$, and $\gamma_{xy}$. Such a situation occurs in a plate subjected along its edges to uniformly distributed loads and restrained from expanding or contracting laterally by smooth, rigid, and fixed supports (Fig. 7.54). It would also be found in a bar of infinite length subjected on its sides to uniformly distributed loads since, by reason of symmetry, the elements located in a given transverse plane cannot move out of that plane. This idealized model shows that, in the actual case of a long bar subjected to uniformly distributed transverse loads (Fig. 7.55), a state of plane strain exists in any given transverse section that is not located too close to either end of the bar.†

Let us assume that a state of plane strain exists at point $Q$ (with $\varepsilon_z = \gamma_{zx} = \gamma_{zy} = 0$), and that it is defined by the strain components $\varepsilon_x$, $\varepsilon_y$, and $\gamma_{xy}$ associated with the $x$ and $y$ axes. As we know from Secs. 2.12 and 2.14, this means that a square element of center $Q$, with sides of length $\Delta s$ respectively parallel to the $x$ and $y$ axes, is deformed into a parallelogram with sides of length respectively equal to $\Delta s (1 + \varepsilon_x)$ and $\Delta s (1 + \varepsilon_y)$, forming angles of $\pi/2 - \gamma_{xy}$ and $\pi/2 + \gamma_{xy}$ with each other (Fig. 7.56). We recall that, as a result of the deformations of the other elements located in the $xy$ plane, the element considered may also undergo a rigid-body motion, but such a motion is irrelevant to the determination of the strains at point $Q$ and will be ignored in this analysis. Our purpose is to determine in terms of $\varepsilon_x$, $\varepsilon_y$, $\gamma_{xy}$, and $\theta$ the strain components $\varepsilon'_x$, $\varepsilon'_y$, and $\gamma_{x'y'}$ associated with the frame of reference $x'y'$ obtained by rotating the $x$ and $y$ axes through the angle $\theta$. As shown in Fig. 7.57, these new strain

†It should be observed that a state of plane strain and a state of plane stress (cf. Sec. 7.1) do not occur simultaneously, except for ideal materials with a Poisson ratio equal to zero. The constraints placed on the elements of the plate of Fig. 7.54 and of the bar of Fig. 7.55 result in a stress $\sigma_y$ different from zero. On the other hand, in the case of the plate of Fig. 7.3, the absence of any lateral restraint results in $\sigma_y = 0$ and $\varepsilon_x \neq 0$. 

---

**Fig. 7.54** Plane strain example: laterally restrained plate.

**Fig. 7.55** Plane strain example: bar of infinite length.

**Fig. 7.56** Plane strain element deformation.

**Fig. 7.57** Transformation of plane strain element.
components define the parallelogram into which a square with sides respectively parallel to the \( x' \) and \( y' \) axes is deformed.

We first derive an expression for the normal strain \( \epsilon(\theta) \) along a line \( AB \) forming an arbitrary angle \( \theta \) with the \( x \) axis. To do so, we consider the right triangle \( ABC \), which has \( AB \) for hypotenuse (Fig. 7.58a), and the oblique triangle \( A'B'C' \) into which triangle \( ABC \) is deformed (Fig. 7.58b). Denoting by \( \Delta s \) the length of \( AB \), we express the lengths of \( A'B' \) as \( \Delta s + [1 + \epsilon(\theta)] \). Similarly, denoting by \( \Delta x \) and \( \Delta y \) the lengths of sides \( AC \) and \( CB \), we express the lengths of \( A'C' \) and \( C'B' \) as \( \Delta x (1 + \epsilon_x) \) and \( \Delta y (1 + \epsilon_y) \), respectively. Recalling from Fig. 7.56 that the right angle at \( C \) in Fig. 7.58a deforms into an angle equal to \( \frac{\pi}{2} + \gamma_{xy} \) in Fig. 7.58b, and applying the law of cosines to triangle \( A'B'C' \), we write

\[
(A'B')^2 = (A'C')^2 + (C'B')^2 - 2(A'C')(C'B') \cos\left(\frac{\pi}{2} + \gamma_{xy}\right)
\]

\[
(\Delta x)^2(1 + \epsilon_x)^2 + (\Delta y)^2(1 + \epsilon_y)^2 - 2(\Delta x)(1 + \epsilon_x)(\Delta y)(1 + \epsilon_y) \cos\left(\frac{\pi}{2} + \gamma_{xy}\right)
\]

(7.38)

But from Fig. 7.58a we have

\[
\Delta x = (\Delta s) \cos \theta \quad \Delta y = (\Delta s) \sin \theta
\]

and we note that, since \( \gamma_{xy} \) is very small,

\[
\cos\left(\frac{\pi}{2} + \gamma_{xy}\right) \approx -\sin \gamma_{xy} \approx -\gamma_{xy}
\]

(7.40)

Substituting from Eqs. (7.39) and (7.40) into Eq. (7.38), recalling that \( \cos^2 \theta + \sin^2 \theta = 1 \), and neglecting second-order terms in \( \epsilon(\theta) \), \( \epsilon_x \), \( \epsilon_y \), and \( \gamma_{xy} \), we write

\[
\epsilon(\theta) = \epsilon_x \cos^2 \theta + \epsilon_y \sin^2 \theta + \gamma_{xy} \sin \theta \cos \theta
\]

(7.41)

Equation (7.41) enables us to determine the normal strain \( \epsilon(\theta) \) in any direction \( AB \) in terms of the strain components \( \epsilon_x \), \( \epsilon_y \), \( \gamma_{xy} \), and the angle \( \theta \) that \( AB \) forms with the \( x \) axis. We check that, for \( \theta = 0 \), Eq. (7.41) yields \( \epsilon(0) = \epsilon_x \) and that, for \( \theta = 90^\circ \), it yields \( \epsilon(90^\circ) = \epsilon_y \). On the other hand, making \( \theta = 45^\circ \) in Eq. (7.41), we obtain the normal strain in the direction of the bisector \( OB \) of the angle formed by the \( x \) and \( y \) axes (Fig. 7.59). Denoting this strain by \( \epsilon_{OB} \), we write

\[
\epsilon_{OB} = \epsilon(45^\circ) = \frac{1}{2}(\epsilon_x + \epsilon_y + \gamma_{xy})
\]

(7.42)

Solving Eq. (7.42) for \( \gamma_{xy} \), we have

\[
\gamma_{xy} = 2\epsilon_{OB} - (\epsilon_x + \epsilon_y)
\]

(7.43)

This relation makes it possible to express the shearing strain associated with a given pair of rectangular axes in terms of the normal strains measured along these axes and their bisector. It will play a fundamental role in our present derivation and will also be used in Sec. 7.13 in connection with the experimental determination of shearing strains.
Recalling that the main purpose of this section is to express the strain components associated with the frame of reference \( x'y' \) of Fig. 7.57 in terms of the angle \( \theta \) and the strain components \( \varepsilon_x', \varepsilon_y' \), and \( \gamma_{xy} \) associated with the \( x \) and \( y \) axes, we note that the normal strain \( \varepsilon_x' \) along the \( x' \) axis is given by Eq. (7.41). Using the trigonometric relations (7.3) and (7.4), we write this equation in the alternative form

\[
\varepsilon_x' = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta \quad (7.44)
\]

Replacing \( \theta \) by \( \theta + 90^\circ \), we obtain the normal strain along the \( y' \) axis.

Since \( \cos (2\theta + 180^\circ) = -\cos 2\theta \) and \( \sin (2\theta + 180^\circ) = -\sin 2\theta \), we have

\[
\varepsilon_y' = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta \quad (7.45)
\]

Adding Eqs. (7.44) and (7.45) member to member, we obtain

\[
\varepsilon_x + \varepsilon_y = \varepsilon_x' + \varepsilon_y' \quad (7.46)
\]

Since \( \varepsilon_z = \varepsilon_z' = 0 \), we thus verify in the case of plane strain that the sum of the normal strains associated with a cubic element of material is independent of the orientation of that element.†

Replacing now \( \theta \) by \( \theta + 45^\circ \) in Eq. (7.44), we obtain an expression for the normal strain along the bisector \( OB' \) of the angle formed by the \( x' \) and \( y' \) axes. Since \( \cos (2\theta + 90^\circ) = -\sin 2\theta \) and \( \sin (2\theta + 90^\circ) = \cos 2\theta \), we have

\[
\varepsilon_{OB'} = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{\varepsilon_x - \varepsilon_y}{2} \sin 2\theta + \frac{\gamma_{xy}}{2} \cos 2\theta \quad (7.47)
\]

Writing Eq. (7.43) with respect to the \( x' \) and \( y' \) axes, we express the shearing strain \( \gamma_{x'y'} \) in terms of the normal strains measured along the \( x' \) and \( y' \) axes and the bisector \( OB' \):

\[
\gamma_{x'y'} = 2\varepsilon_{OB'} - (\varepsilon_x' + \varepsilon_y') \quad (7.48)
\]

Substituting from Eqs. (7.46) and (7.47) into (7.48), we obtain

\[
\gamma_{x'y'} = - (\varepsilon_x - \varepsilon_y) \sin 2\theta + \gamma_{xy} \cos 2\theta \quad (7.49)
\]

Equations (7.44), (7.45), and (7.49) are the desired equations defining the transformation of plane strain under a rotation of axes in the plane of strain. Dividing all terms in Eq. (7.49) by 2, we write this equation in the alternative form

\[
\frac{\gamma_{x'y'}}{2} = - \frac{\varepsilon_x - \varepsilon_y}{2} \sin 2\theta + \frac{\gamma_{xy}}{2} \cos 2\theta \quad (7.49')
\]

and observe that Eqs. (7.44), (7.45), and (7.49') for the transformation of plane strain closely resemble the equations derived in Sec. 7.2 for the transformation of plane stress. While the former may be obtained from the latter by replacing the normal stresses by the corresponding normal strains, it should be noted, however, that the shearing stresses \( \tau_{xy} \) and \( \tau_{x'y'} \) should be replaced by half of the corresponding shearing strains, i.e., by \( \frac{1}{2} \gamma_{xy} \) and \( \frac{1}{2} \gamma_{x'y'} \), respectively.

†Cf. first footnote on page 97.
7.11 Mohr’s Circle for Plane Strain

Since the equations for the transformation of plane strain are of the same form as the equations for the transformation of plane stress, the use of Mohr's circle can be extended to the analysis of plane strain. Given the strain components \( \varepsilon_x, \varepsilon_y, \) and \( \gamma_{xy} \) defining the deformation represented in Fig. 7.56, we plot a point \( X(\varepsilon_x, -\frac{1}{2} \gamma_{xy}) \) of abscissa equal to the normal strain \( \varepsilon_x \) and of ordinate equal to minus half the shearing strain \( \gamma_{xy} \) and a point \( Y(\varepsilon_y, +\frac{1}{2} \gamma_{xy}) \) (Fig. 7.60). Drawing the diameter \( XY \), we define the center \( C \) of Mohr’s circle for plane strain. The abscissa of \( C \) and the radius \( R \) of the circle are respectively equal to

\[
\varepsilon_{\text{ave}} = \frac{\varepsilon_x + \varepsilon_y}{2} \quad \text{and} \quad R = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \quad (7.50)
\]

We note that if \( \gamma_{xy} \) is positive, as assumed in Fig. 7.56, points \( X \) and \( Y \) are plotted, respectively, below and above the horizontal axis in Fig. 7.60. But, in the absence of any overall rigid-body rotation, the side of the element in Fig. 7.56 that is associated with \( \varepsilon_x \), observed to rotate counterclockwise, while the side associated with \( \varepsilon_y \) is observed to rotate clockwise. Thus, if the shear deformation causes a given side to rotate clockwise, the corresponding point on Mohr’s circle for plane strain is plotted above the horizontal axis, and if the deformation causes the side to rotate counterclockwise, the corresponding point is plotted below the horizontal axis. It is noted that this convention matches the convention used to draw Mohr’s circle for plane stress.

Points \( A \) and \( B \) where Mohr’s circle intersects the horizontal axis correspond to the principal strains \( \varepsilon_{\text{max}} \) and \( \varepsilon_{\text{min}} \) (Fig. 7.61a). We find

\[
\varepsilon_{\text{max}} = \varepsilon_{\text{ave}} + R \quad \text{and} \quad \varepsilon_{\text{min}} = \varepsilon_{\text{ave}} - R \quad (7.51)
\]

where \( \varepsilon_{\text{ave}} \) and \( R \) are defined by Eqs. (7.50). The corresponding value \( \theta_p \) of the angle \( \theta \) is obtained by observing that the shearing strain is zero for \( A \) and \( B \). Setting \( \gamma_{xy} = 0 \) in Eq. (7.49), we have

\[
\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} \quad (7.52)
\]

The corresponding axes \( a \) and \( b \) in Fig. 7.61b are the principal axes of strain. The angle \( \theta_p \), which defines the direction of the principal axis \( Oa \) in Fig. 7.61b corresponding to point \( A \) in Fig. 7.61a, is equal to half of the angle \( XCA \) measured on Mohr’s circle, and the rotation that brings \( Ox \) into \( Oa \) has the same sense as the rotation that brings the diameter \( XY \) of Mohr’s circle into the diameter \( AB \).

We recall from Sec. 2.14 that, in the case of the elastic deformation of a homogeneous, isotropic material, Hooke’s law for shearing stress and strain applies and yields \( \tau_{xy} = G\gamma_{xy} \) for any pair of rectangular \( x \) and \( y \) axes. Thus, \( \gamma_{xy} = 0 \) when \( \tau_{xy} = 0 \), which indicates that the principal axes of strain coincide with the principal axes of stress.
The maximum in-plane shearing strain is defined by points $D$ and $E$ in Fig. 7.61. It is equal to the diameter of Mohr’s circle. Recalling the second of Eqs. (7.50), we write

$$
\gamma_{\text{max (in plane)}} = 2R = \sqrt{(\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}^2}
$$  (7.53)

Finally, we note that the points $X'$ and $Y'$ that define the components of strain corresponding to a rotation of the coordinate axes through an angle $\theta$ (Fig. 7.57) are obtained by rotating the diameter $XY$ of Mohr’s circle in the same sense through an angle $2\theta$ (Fig. 7.62).

**EXAMPLE 7.04** In a material in a state of plane strain, it is known that the horizontal side of a $10 \times 10$-mm square elongates by $4 \mu$m, while its vertical side remains unchanged, and that the angle at the lower left corner increases by $0.4 \times 10^{-3}$ rad (Fig. 7.63). Determine (a) the principal axes and principal strains, (b) the maximum shearing strain and the corresponding normal strain.

**a) Principal Axes and Principal Strains.** We first determine the coordinates of points $X$ and $Y$ on Mohr’s circle for strain. We have

$$
\begin{align*}
\varepsilon_x &= +4 \times 10^{-6} \text{ m} = +400 \mu \\
\varepsilon_y &= 0 \\
\gamma_{xy} &= 200 \mu
\end{align*}
$$

Since the side of the square associated with $\varepsilon_x$ rotates clockwise, point $X$ of coordinates $\varepsilon_x$ and $\varepsilon_y / 2$ is plotted above the horizontal axis. Since $\varepsilon_y = 0$ and the corresponding side rotates counterclockwise, point $Y$ is plotted directly below the origin (Fig. 7.64). Drawing the diameter $XY$, we determine the center $C$ of Mohr’s circle and its radius $R$. We have

$$
OC = \frac{\varepsilon_x + \varepsilon_y}{2} = 200 \mu \\
R = \sqrt{(OC)^2 + (OY)^2} = \sqrt{(200 \mu)^2 + (200 \mu)^2} = 283 \mu
$$

The principal strains are defined by the abscissas of points $A$ and $B$. We write

$$
\varepsilon_x = OA = OC + R = 200 \mu + 283 \mu = 483 \mu \\
\varepsilon_y = OB = OC - R = 200 \mu - 283 \mu = -83 \mu
$$

The principal axes $OA$ and $OB$ are shown in Fig. 7.65. Since $OC = OY$, the angle at $C$ in triangle $OCY$ is $45^\circ$. Thus, the angle $2\theta_p$ that brings $XY$ into $AB$ is $45^\circ$ and the angle $\theta_p$ bringing $Ox$ into $OA$ is $22.5^\circ$. 

**b) Maximum Shearing Strain.** We determine the coordinates of points $X$ and $Y$ on Mohr’s circle for strain. We have

$$
\begin{align*}
\varepsilon_x &= +4 \times 10^{-6} \text{ m} = +400 \mu \\
\varepsilon_y &= 0 \\
\gamma_{xy} &= 200 \mu
\end{align*}
$$

Since the side of the square associated with $\varepsilon_x$ rotates clockwise, point $X$ of coordinates $\varepsilon_x$ and $\varepsilon_y / 2$ is plotted above the horizontal axis. Since $\varepsilon_y = 0$ and the corresponding side rotates counterclockwise, point $Y$ is plotted directly below the origin (Fig. 7.64). Drawing the diameter $XY$, we determine the center $C$ of Mohr’s circle and its radius $R$. We have

$$
OC = \frac{\varepsilon_x + \varepsilon_y}{2} = 200 \mu \\
R = \sqrt{(OC)^2 + (OY)^2} = \sqrt{(200 \mu)^2 + (200 \mu)^2} = 283 \mu
$$

The principal strains are defined by the abscissas of points $A$ and $B$. We write

$$
\varepsilon_x = OA = OC + R = 200 \mu + 283 \mu = 483 \mu \\
\varepsilon_y = OB = OC - R = 200 \mu - 283 \mu = -83 \mu
$$

The principal axes $OA$ and $OB$ are shown in Fig. 7.65. Since $OC = OY$, the angle at $C$ in triangle $OCY$ is $45^\circ$. Thus, the angle $2\theta_p$ that brings $XY$ into $AB$ is $45^\circ$ and the angle $\theta_p$ bringing $Ox$ into $OA$ is $22.5^\circ$. 

**EXAMPLE 7.04** In a material in a state of plane strain, it is known that the horizontal side of a $10 \times 10$-mm square elongates by $4 \mu$m, while its vertical side remains unchanged, and that the angle at the lower left corner increases by $0.4 \times 10^{-3}$ rad (Fig. 7.63). Determine (a) the principal axes and principal strains, (b) the maximum shearing strain and the corresponding normal strain.
We saw in Sec. 7.5 that, in the most general case of stress, we can determine three coordinate axes \( a, b, \) and \( c \), called the principal axes of stress. A small cubic element with faces respectively perpendicular to these axes is free of shearing stresses (Fig. 7.25); i.e., we have

\[
\gamma_{ab} = \gamma_{bc} = \gamma_{ca} = 0.
\]

As recalled in the preceding section, Hooke’s law for shearing stress and strain applies when the deformation is elastic and the material homogeneous and isotropic. It follows that, in such a case, the axes \( a, b, \) and \( c \) are also principal axes of strain. A small cube of side equal to unity, centered at \( O \) and with faces respectively perpendicular to the principal axes, is deformed into a rectangular parallelepiped of sides \( 1 + \epsilon_a, 1 + \epsilon_b, \) and \( 1 + \epsilon_c \) (Fig. 7.67).

### (b) Maximum Shearing Strain.

Points \( D \) and \( E \) define the maximum in-plane shearing strain which, since the principal strains have opposite signs, is also the actual maximum shearing strain (see Sec. 7.12). We have

\[
\frac{\gamma_{\text{max}}}{2} = R = 283 \mu \quad \gamma_{\text{max}} = 566 \mu
\]

The corresponding normal strains are both equal to

\[
\epsilon' = OC = 200 \mu
\]

The axes of maximum shearing strain are shown in Fig. 7.66.
If the element of Fig. 7.67 is rotated about one of the principal axes at Q, say the c axis (Fig. 7.68), the method of analysis developed earlier for the transformation of plane strain can be used to determine the strain components $\varepsilon_x$, $\varepsilon_y$, and $\gamma_{xy}$ associated with the faces perpendicular to the c axis, since the derivation of this method did not involve any of the other strain components.† We can, therefore, draw Mohr's circle through the points A and B corresponding to the principal axes a and b (Fig. 7.69). Similarly, circles of diameters BC and CA can be used to analyze the transformation of strain as the element is rotated about the a and b axes, respectively.

The three-dimensional analysis of strain by means of Mohr's circle is limited here to rotations about principal axes (as was the case for the analysis of stress) and is used to determine the maximum shearing strain $\gamma_{\text{max}}$ at point Q. Since $\gamma_{\text{max}}$ is equal to the diameter of the largest of the three circles shown in Fig. 7.69, we have

$$\gamma_{\text{max}} = \frac{1}{2} \gamma_{\text{max}}$$

(7.54)

where $\epsilon_{\text{max}}$ and $\epsilon_{\text{min}}$ represent the \textit{algebraic} values of the maximum and minimum strains at point Q.

†We note that the other four faces of the element remain rectangular and that the edges parallel to the c axis remain unchanged.
Returning to the particular case of plane strain, and selecting the x and y axes in the plane of strain, we have $\varepsilon_z = \gamma_{zx} = \gamma_{zy} = 0$. Thus, the z axis is one of the three principal axes at Q, and the corresponding point in the Mohr-circle diagram is the origin O, where $\varepsilon = \gamma = 0$. If the points A and B that define the principal axes within the plane of strain fall on opposite sides of O (Fig. 7.70(a)), the corresponding principal strains represent the maximum and minimum normal strains at point Q, and the maximum shearing strain is equal to the maximum in-plane shearing strain corresponding to points D and E. If, on the other hand, A and B are on the same side of O (Fig. 7.70(b)), that is, if $\varepsilon_a$ and $\varepsilon_b$ have the same sign, then the maximum shearing strain is defined by points $D'$ and $E'$ on the circle of diameter OA, and we have $\gamma_{\text{max}} = \varepsilon_{\text{max}}$.

We now consider the particular case of plane stress encountered in a thin plate or on the free surface of a structural element or machine component (Sec. 7.1). Selecting the x and y axes in the plane of stress, we have $s_z = t_{zx} = t_{zy} = 0$ and verify that the z axis is a principal axis of stress. As we saw earlier, if the deformation is elastic and if the material is homogeneous and isotropic, it follows from Hooke’s law that $\gamma_{zx} = \gamma_{zy} = 0$; thus, the z axis is also a principal axis of strain, and Mohr’s circle can be used to analyze the transformation of strain in the $xy$ plane. However, as we shall see presently, it does not follow from Hooke’s law that $\varepsilon_z = 0$; indeed, a state of plane stress does not, in general, result in a state of plane strain.†

Denoting by $a$ and $b$ the principal axes within the plane of stress, and by $c$ the principal axis perpendicular to that plane, we let $\sigma_a = s_a$, $\sigma_b = s_b$, and $\sigma_c = s_z$ in Eq. (2.24) for the generalized Hooke’s law (Sec. 2.12) and write

$$\varepsilon_a = \frac{\sigma_a}{E} - \frac{\nu \sigma_b}{E}$$  \hspace{1cm} (7.55)

$$\varepsilon_b = -\frac{\nu \sigma_a}{E} + \frac{\sigma_b}{E}$$  \hspace{1cm} (7.56)

$$\varepsilon_c = -\frac{\nu}{E} (\sigma_a + \sigma_b)$$  \hspace{1cm} (7.57)

Adding Eqs. (7.55) and (7.56) member to member, we have

$$\varepsilon_a + \varepsilon_b = \frac{1 - \nu}{E} (\sigma_a + \sigma_b)$$  \hspace{1cm} (7.58)

Solving Eq. (7.58) for $\sigma_a + \sigma_b$ and substituting into Eq. (7.57), we write

$$\varepsilon_c = -\frac{\nu}{1 - \nu} (\varepsilon_a + \varepsilon_b)$$  \hspace{1cm} (7.59)

The relation obtained defines the third principal strain in terms of the “in-plane” principal strains. We note that, if B is located between A and C on the Mohr-circle diagram (Fig. 7.71), the maximum shearing strain is equal to the diameter CA of the circle corresponding to a rotation about the b axis, out of the plane of stress.

†See footnote on page 486.
As a result of measurements made on the surface of a machine component with strain gages oriented in various ways, it has been established that the principal strains on the free surface are $\varepsilon_a = +400 \times 10^{-6}$ in./in. and $\varepsilon_b = -50 \times 10^{-6}$ in./in. Knowing that Poisson's ratio for the given material is $\nu = 0.30$, determine (a) the maximum in-plane shearing strain, (b) the true value of the maximum shearing strain near the surface of the component.

(a) Maximum In-Plane Shearing Strain. We draw Mohr's circle through the points A and B corresponding to the given principal strains (Fig. 7.72). The maximum in-plane shearing strain is defined by points D and E and is equal to the diameter of Mohr's circle:

$$\gamma_{\text{max (in plane)}} = 400 \times 10^{-6} + 50 \times 10^{-6} = 450 \times 10^{-6} \text{ rad}$$

(b) Maximum Shearing Strain. We first determine the third principal strain $\varepsilon_c$. Since we have a state of plane stress on the surface of the machine component, we use Eq. (7.59) and write

$$\varepsilon_c = -\frac{\nu}{1 - \nu} (\varepsilon_a + \varepsilon_b) = -\frac{0.30}{0.70} (400 \times 10^{-6} - 50 \times 10^{-6}) = -150 \times 10^{-6} \text{ in./in.}$$

Drawing Mohr's circles through A and C and through B and C (Fig. 7.73), we find that the maximum shearing strain is equal to the diameter of the circle of diameter CA:

$$\gamma_{\text{max}} = 400 \times 10^{-6} + 150 \times 10^{-6} = 550 \times 10^{-6} \text{ rad}$$

We note that, even though $\varepsilon_a$ and $\varepsilon_b$ have opposite signs, the maximum in-plane shearing strain does not represent the true maximum shearing strain.
load has been applied. If \( L \) is the undeformed length of \( AB \) and \( \delta \) its deformation, the normal strain along \( AB \) is \( \varepsilon_{AB} = \delta/L \).

A more convenient and more accurate method for the measurement of normal strains is provided by electrical strain gages. A typical electrical strain gage consists of a length of thin wire arranged as shown in Fig. 7.74 and cemented to two pieces of paper. In order to measure the strain \( \varepsilon_{AB} \) of a given material in the direction \( AB \), the gage is cemented to the surface of the material, with the wire folds running parallel to \( AB \). As the material elongates, the wire increases in length and decreases in diameter, causing the electrical resistance of the gage to increase. By measuring the current passing through a properly calibrated gage, the strain \( \varepsilon_{AB} \) can be determined accurately and continuously as the load is increased.

The strain components \( \varepsilon_x \) and \( \varepsilon_y \) can be determined at a given point of the free surface of a material by simply measuring the normal strain along \( x \) and \( y \) axes drawn through that point. Recalling Eq. (7.43) of Sec. 7.10, we note that a third measurement of normal strain, made along the bisector \( OB \) of the angle formed by the \( x \) and \( y \) axes, enables us to determine the shearing strain \( \gamma_{xy} \) as well (Fig. 7.75):

\[
\gamma_{xy} = 2\varepsilon_{OB} - (\varepsilon_x + \varepsilon_y)
\]

(7.43)

It should be noted that the strain components \( \varepsilon_x \), \( \varepsilon_y \), and \( \gamma_{xy} \) at a given point could be obtained from normal strain measurements made along any three lines drawn through that point (Fig. 7.76). Denoting respectively by \( \theta_1 \), \( \theta_2 \), and \( \theta_3 \) the angle each of the three lines forms with the \( x \) axis, by \( \varepsilon_1 \), \( \varepsilon_2 \), and \( \varepsilon_3 \) the corresponding strain measurements, and substituting into Eq. (7.41), we write the three equations:

\[
\begin{align*}
\varepsilon_1 &= \varepsilon_x \cos^2 \theta_1 + \varepsilon_y \sin^2 \theta_1 + \gamma_{xy} \sin \theta_1 \cos \theta_1 \\
\varepsilon_2 &= \varepsilon_x \cos^2 \theta_2 + \varepsilon_y \sin^2 \theta_2 + \gamma_{xy} \sin \theta_2 \cos \theta_2 \\
\varepsilon_3 &= \varepsilon_x \cos^2 \theta_3 + \varepsilon_y \sin^2 \theta_3 + \gamma_{xy} \sin \theta_3 \cos \theta_3
\end{align*}
\]

(7.60)

which can be solved simultaneously for \( \varepsilon_x \), \( \varepsilon_y \), and \( \gamma_{xy} \).†

The arrangement of strain gages used to measure the three normal strains \( \varepsilon_1 \), \( \varepsilon_2 \), and \( \varepsilon_3 \) is known as a strain rosette. The rosette used to measure normal strains along the \( x \) and \( y \) axes and their bisector is referred to as a 45° rosette (Fig. 7.75). Another rosette frequently used is the 60° rosette (see Sample Prob. 7.7).

†It should be noted that the free surface on which the strain measurements are made is in a state of plane stress, while Eqs. (7.41) and (7.43) were derived for a state of plane strain. However, as observed earlier, the normal to the free surface is a principal axis of strain and the derivations given in Sec. 7.10 remain valid.
SAMPLE PROBLEM 7.6

A cylindrical storage tank used to transport gas under pressure has an inner diameter of 24 in. and a wall thickness of \( \frac{3}{4} \) in. Strain gages attached to the surface of the tank in transverse and longitudinal directions indicate strains of \( 255 \times 10^{-6} \) and \( 60 \times 10^{-6} \) in./in. respectively. Knowing that a torsion test has shown that the modulus of rigidity of the material used in the tank is \( G = 11.2 \times 10^6 \) psi, determine (a) the gage pressure inside the tank, (b) the principal stresses and the maximum shearing stress in the wall of the tank.

**SOLUTION**

a. Gage Pressure Inside Tank. We note that the given strains are the principal strains at the surface of the tank. Plotting the corresponding points A and B, we draw Mohr’s circle for strain. The maximum in-plane shearing strain is equal to the diameter of the circle.

\[
\frac{\gamma_{\text{max}}}{2} = \frac{\epsilon_1 - \epsilon_2}{2} = 255 \times 10^{-6} - 60 \times 10^{-6} = 195 \times 10^{-6} \text{ rad}
\]

From Hooke’s law for shearing stress and strain, we have

\[
\tau_{\text{max (in plane)}} = G \gamma_{\text{max (in plane)}} = (11.2 \times 10^6 \text{ psi})(195 \times 10^{-6} \text{ rad}) = 2184 \text{ psi} = 2.184 \text{ ksi}
\]

Substituting this value and the given data in Eq. (7.33), we write

\[
\tau_{\text{max (in plane)}} = \frac{pr}{4t} \quad 2184 \text{ psi} = \frac{p(12 \text{ in.})}{4(0.75 \text{ in.})}
\]

Solving for the gage pressure \( p \), we have

\[
p = 546 \text{ psi}
\]

b. Principal Stresses and Maximum Shearing Stress. Recalling that, for a thin-walled cylindrical pressure vessel, \( \sigma_1 = 2\sigma_2 \), we draw Mohr’s circle for stress and obtain

\[
\sigma_2 = \frac{2\tau_{\text{max (in plane)}}}{2} = 2(2.184 \text{ ksi}) = 4.368 \text{ ksi} \quad \sigma_2 = 4.37 \text{ ksi}
\]

\[
\sigma_1 = 2\sigma_2 = 2(4.368 \text{ ksi}) \quad \sigma_1 = 8.74 \text{ ksi}
\]

The maximum shearing stress is equal to the radius of the circle of diameter OA and corresponds to a rotation of 45° about a longitudinal axis.

\[
\tau_{\text{max}} = \frac{1}{2} \sigma_1 = \sigma_2 = 4.368 \text{ ksi} \quad \tau_{\text{max}} = 4.37 \text{ ksi}
\]
SAMPLE PROBLEM 7.7

Using a 60° rosette, the following strains have been determined at point Q on the surface of a steel machine base:

\[ \epsilon_1 = 40 \mu \quad \epsilon_2 = 980 \mu \quad \epsilon_3 = 330 \mu \]

Using the coordinate axes shown, determine at point Q, (a) the strain components \( \epsilon_x, \epsilon_y \) and \( \gamma_{xy} \) (b) the principal strains, (c) the maximum shearing strain. (Use \( \nu = 0.29 \).)

SOLUTION

a. Strain Components \( \epsilon_x, \epsilon_y, \gamma_{xy} \)

For the coordinate axes shown

\[ \theta_1 = 0 \quad \theta_2 = 60^\circ \quad \theta_3 = 120^\circ \]

Substituting these values into Eqs. (7.60), we have

\[ \epsilon_1 = \epsilon_x(1) + \epsilon_y(0) + \gamma_{xy}(0)(1) \]
\[ \epsilon_2 = \epsilon_x(0.500)^2 + \epsilon_y(0.866)^2 + \gamma_{xy}(0.866)(0.500) \]
\[ \epsilon_3 = \epsilon_x(-0.500)^2 + \epsilon_y(0.866)^2 + \gamma_{xy}(0.866)(-0.500) \]

Solving these equations for \( \epsilon_x, \epsilon_y, \) and \( \gamma_{xy} \) we obtain

\[ \epsilon_x = \epsilon_1 \quad \epsilon_y = \frac{1}{2}[2\epsilon_2 + 2\epsilon_3 - \epsilon_1] \quad \gamma_{xy} = \frac{\epsilon_2 - \epsilon_3}{0.866} \]

Substituting the given values for \( \epsilon_1, \epsilon_2, \) and \( \epsilon_3, \) we have

\[ \epsilon_x = 40 \mu \quad \epsilon_y = \frac{1}{2}[2(980) + 2(330) - 40] \quad \gamma_{xy} = +860 \mu \]
\[ \gamma_{xy} = (980 - 330)/0.866 \]

These strains are indicated on the element shown.

b. Principal Strains.

We note that the side of the element associated with \( \epsilon_x \) rotates counterclockwise; thus, we plot point X below the horizontal axis, i.e., X(40, -375). We then plot Y(860, +375) and draw Mohr’s circle.

\[ \epsilon_{ave} = \frac{1}{3}(860 \mu + 40 \mu) = 450 \mu \]
\[ R = \sqrt{(375 \mu)^2 + (410 \mu)^2} = 556 \mu \]
\[ \tan 2\theta_p = \frac{375 \mu}{410 \mu} \quad 2\theta_p = 42.4^\circ \quad \theta_p = 21.2^\circ \]

Points A and B correspond to the principal strains. We have

\[ \epsilon_a = \epsilon_{ave} - R = 450 \mu - 556 \mu \quad \epsilon_a = -106 \mu \]
\[ \epsilon_b = \epsilon_{ave} + R = 450 \mu + 556 \mu \quad \epsilon_b = +1066 \mu \]

Since \( \sigma_z = 0 \) on the surface, we use Eq. (7.59) to find the principal strain \( \epsilon_c: \)

\[ \epsilon_c = -\frac{\nu}{1-\nu} (\epsilon_a + \epsilon_b) = -\frac{0.29}{1-0.29} (-106 \mu + 1066 \mu) \quad \epsilon_c = -368 \mu \]

\[ \gamma_{max} = \frac{1}{2}(1006 \mu + 368 \mu) \quad \gamma_{max} = 1374 \mu \]
**PROBLEMS**

### 7.128 through 7.131
For the given state of plane strain, use the method of Sec. 7.10 to determine the state of plane strain associated with axes $x'$ and $y'$ rotated through the given angle $\theta$.

<table>
<thead>
<tr>
<th>$\epsilon_x$</th>
<th>$\epsilon_y$</th>
<th>$\gamma_{xy}$</th>
<th>$\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-500\mu$</td>
<td>$+250\mu$</td>
<td>0</td>
<td>$15^\circ$</td>
</tr>
<tr>
<td>$+240\mu$</td>
<td>$+160\mu$</td>
<td>$+150\mu$</td>
<td>$60^\circ$</td>
</tr>
<tr>
<td>$-800\mu$</td>
<td>$+450\mu$</td>
<td>$+200\mu$</td>
<td>$25^\circ$</td>
</tr>
<tr>
<td>0</td>
<td>$+320\mu$</td>
<td>$-100\mu$</td>
<td>$30^\circ$</td>
</tr>
</tbody>
</table>

### 7.132 through 7.135
For the given state of plane strain, use Mohr’s circle to determine the state of plane strain associated with axes $x'$ and $y'$ rotated through the given angle $\theta$.

### 7.136 through 7.139
The following state of strain has been measured on the surface of a thin plate. Knowing that the surface of the plate is unstressed, determine (a) the direction and magnitude of the principal strains, (b) the maximum in-plane shearing strain, (c) the maximum shearing strain. (Use $\nu = \frac{1}{3}$)

<table>
<thead>
<tr>
<th>$\epsilon_x$</th>
<th>$\epsilon_y$</th>
<th>$\gamma_{xy}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-260\mu$</td>
<td>$-60\mu$</td>
<td>$+480\mu$</td>
</tr>
<tr>
<td>$-600\mu$</td>
<td>$-400\mu$</td>
<td>$+350\mu$</td>
</tr>
<tr>
<td>$+160\mu$</td>
<td>$-480\mu$</td>
<td>$-600\mu$</td>
</tr>
<tr>
<td>$+30\mu$</td>
<td>$+570\mu$</td>
<td>$+720\mu$</td>
</tr>
</tbody>
</table>

### 7.140 through 7.143
For the given state of plane strain, use Mohr’s circle to determine (a) the orientation and magnitude of the principal strains, (b) the maximum in-plane strain, (c) the maximum shearing strain.

<table>
<thead>
<tr>
<th>$\epsilon_x$</th>
<th>$\epsilon_y$</th>
<th>$\gamma_{xy}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$+60\mu$</td>
<td>$+240\mu$</td>
<td>$-50\mu$</td>
</tr>
<tr>
<td>$+400\mu$</td>
<td>$+200\mu$</td>
<td>$+375\mu$</td>
</tr>
<tr>
<td>$+300\mu$</td>
<td>$+60\mu$</td>
<td>$+100\mu$</td>
</tr>
<tr>
<td>$-180\mu$</td>
<td>$-260\mu$</td>
<td>$+315\mu$</td>
</tr>
</tbody>
</table>

### 7.144
Determine the strain $\epsilon_i$ knowing that the following strains have been determined by use of the rosette shown:

$$\epsilon_1 = +480\mu, \quad \epsilon_2 = -120\mu, \quad \epsilon_3 = +80\mu$$
7.145 The strains determined by the use of the rosette shown during the test of a machine element are
\[ \epsilon_1 = +600 \mu \quad \epsilon_2 = +450 \mu \quad \epsilon_3 = -75 \mu \]
Determine (a) the in-plane principal strains, (b) the in-plane maximum shearing strain.

7.146 The rosette shown has been used to determine the following strains at a point on the surface of a crane hook:
\[ \epsilon_1 = +420 \times 10^{-6} \text{ in./in.} \quad \epsilon_2 = -45 \times 10^{-6} \text{ in./in.} \quad \epsilon_4 = +165 \times 10^{-6} \text{ in./in.} \]
(a) What should be the reading of gage 3? (b) Determine the principal strains and the maximum in-plane shearing strain.

7.147 The strains determined by the use of the rosette attached as shown during the test of a machine element are
\[ \epsilon_1 = -93.1 \times 10^{-6} \text{ in./in.} \quad \epsilon_3 = +135 \times 10^{-6} \text{ in./in.} \]
Determine (a) the orientation and magnitude of the principal strains in the plane of the rosette, (b) the maximum in-plane shearing strain.

7.148 Using a 45° rosette, the strains \( \epsilon_1 \), \( \epsilon_2 \) and \( \epsilon_3 \) have been determined at a given point. Using Mohr's circle, show that the principal strains are:
\[ \epsilon_{\text{max, min}} = \frac{1}{2} (\epsilon_1 + \epsilon_3) \pm \frac{1}{\sqrt{2}} \left[ (\epsilon_1 - \epsilon_3)^2 + (\epsilon_2 - \epsilon_3)^2 \right]^{1/2} \]
(Hint: The shaded triangles are congruent.)
Show that the sum of the three strain measurements made with a 60° rosette is independent of the orientation of the rosette and equal to

\[ \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 3\varepsilon_{avg} \]

where \( \varepsilon_{avg} \) is the abscissa of the center of the corresponding Mohr’s circle.

7.150 A single strain gage is cemented to a solid 4-in.-diameter steel shaft at an angle \( \beta = 25° \) with a line parallel to the axis of the shaft. Knowing that \( G = 11.5 \times 10^6 \) psi, determine the torque \( T \) indicated by a gage reading of \( 300 \times 10^{-6} \) in./in.

7.151 Solve Prob. 7.150, assuming that the gage forms an angle \( \beta = 35° \) with a line parallel to the axis of the shaft.

7.152 A single strain gage forming an angle \( \beta = 18° \) with a horizontal plane is used to determine the gage pressure in the cylindrical steel tank shown. The cylindrical wall of the tank is 6-mm thick, has a 600-mm inside diameter, and is made of a steel with \( E = 200 \) GPa and \( \nu = 0.30 \). Determine the pressure in the tank indicated by a strain gage reading of \( 280\mu \).

7.153 Solve Prob. 7.152, assuming that the gage forms an angle \( \beta = 35° \) with a horizontal plane.
7.154 The given state of plane stress is known to exist on the surface of a machine component. Knowing that $E = 200$ GPa and $G = 77.2$ GPa, determine the direction and magnitude of the three principal strains $(a)$ by determining the corresponding state of strain [use Eq. (2.43) and Eq. (2.38)] and then using Mohr’s circle for strain, $(b)$ by using Mohr’s circle for stress to determine the principal planes and principal stresses and then determining the corresponding strains.

7.155 The following state of strain has been determined on the surface of a cast-iron machine part:

$$\epsilon_x = -720 \mu \quad \epsilon_y = -400 \mu \quad \gamma_{xy} = +660 \mu$$

Knowing that $E = 69$ GPa and $G = 28$ GPa, determine the principal planes and principal stresses $(a)$ by determining the corresponding state of plane stress [use Eq. (2.36), Eq. (2.43), and the first two equations of Prob. 2.72] and then using Mohr’s circle for stress, $(b)$ by using Mohr’s circle for strain to determine the orientation and magnitude of the principal strains and then determine the corresponding stresses.

7.156 A centric axial force $P$ and a horizontal force $Q_x$ are both applied at point $C$ of the rectangular bar shown. A $45^\circ$ strain rosette on the surface of the bar at point A indicates the following strains:

$$\epsilon_1 = -60 \times 10^{-6} \text{ in./in.} \quad \epsilon_2 = +240 \times 10^{-6} \text{ in./in.} \quad \epsilon_3 = +200 \times 10^{-6} \text{ in./in.}$$

Knowing that $E = 29 \times 10^6$ psi and $\nu = 0.30$, determine the magnitudes of $P$ and $Q_x$.
The first part of this chapter was devoted to a study of the transformation of stress under a rotation of axes and to its application to the solution of engineering problems, and the second part to a similar study of the transformation of strain.

**Transformation of plane stress**

Considering first a state of plane stress at a given point \( Q \) [Sec. 7.2] and denoting by \( \sigma_x, \sigma_y \), and \( \tau_{xy} \) the stress components associated with the element shown in Fig. 7.77\( a \), we derived the following formulas defining the components \( \sigma_{x'}, \sigma_{y'}, \) and \( \tau_{x'y'} \) associated with that element after it had been rotated through an angle \( \theta \) about the \( z \) axis (Fig. 7.77\( b \)):

\[
\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \quad (7.5)
\]
\[
\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta \quad (7.7)
\]
\[
\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta \quad (7.6)
\]

In Sec. 7.3, we determined the values \( \theta_p \) of the angle of rotation which correspond to the maximum and minimum values of the normal stress at point \( Q \). We wrote

\[
\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \quad (7.12)
\]

**Principal planes. Principal stresses**

The two values obtained for \( \theta_p \) are 90° apart (Fig. 7.78) and define the principal planes of stress at point \( Q \). The corresponding values
of the normal stress are called the principal stresses at $Q$; we obtained

$$
\sigma_{\text{max, min}} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{7.14}
$$

We also noted that the corresponding value of the shearing stress is zero. Next, we determined the values $\theta_1$ of the angle $\theta$ for which the largest value of the shearing stress occurs. We wrote

$$
\tan 2\theta_1 = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}} \tag{7.15}
$$

The two values obtained for $\theta_1$ are $90^\circ$ apart (Fig. 7.79). We also noted that the planes of maximum shearing stress are at $45^\circ$ to the principal planes. The maximum value of the shearing stress for a rotation in the plane of stress is

$$
\tau_{\text{max}} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{7.16}
$$

and the corresponding value of the normal stresses is

$$
\sigma' = \sigma_{\text{ave}} = \frac{\sigma_x + \sigma_y}{2} \tag{7.17}
$$

We saw in Sec. 7.4 that Mohr’s circle provides an alternative method, based on simple geometric considerations, for the analysis of the transformation of plane stress. Given the state of stress shown in black in Fig. 7.80a, we plot point $X$ of coordinates $\sigma_x - \tau_{xy}$ and point $Y$ of coordinates $\sigma_y + \tau_{xy}$ (Fig. 7.80b). Drawing the circle of diameter $XY$, we obtain Mohr’s circle. The abscissas of the points of intersection $A$ and $B$ of the circle with the horizontal axis represent the principal stresses, and the angle of rotation bringing the diameter $XY$ into $AB$ is twice the angle $\theta_1$ defining the principal planes in Fig. 7.80a, with both angles having the same sense. We also noted that diameter $DE$ defines the maximum shearing stress and the orientation of the corresponding plane (Fig. 7.81) [Example 7.02, Sample Probs. 7.2 and 7.3].
Transformations of Stress and Strain

General state of stress

Considering a general state of stress characterized by six stress components [Sec. 7.5], we showed that the normal stress on a plane of arbitrary orientation can be expressed as a quadratic form of the direction cosines of the normal to that plane. This proves the existence of three principal axes of stress and three principal stresses at any given point. Rotating a small cubic element about each of the three principal axes [Sec. 7.6], we drew the corresponding Mohr's circles that yield the values of $\sigma_{\text{max}}$, $\sigma_{\text{min}}$, and $\tau_{\text{max}}$ (Fig. 7.82). In the particular case of plane stress, and if the $x$ and $y$ axes are selected in the plane of stress, point C coincides with the origin $O$. If $A$ and $B$ are located on opposite sides of $O$, the maximum shearing stress is equal to the maximum "in-plane" shearing stress as determined in Secs. 7.3 or 7.4. If $A$ and $B$ are located on the same side of $O$, this will not be the case. If $\sigma_a > \sigma_b > 0$, for instance the maximum shearing stress is equal to $\frac{1}{2} \sigma_a$ and corresponds to a rotation out of the plane of stress (Fig. 7.83).

Yield criteria for ductile materials

Yield criteria for ductile materials under plane stress were developed in Sec. 7.7. To predict whether a structural or machine component will fail at some critical point due to yielding in the material, we first determine the principal stresses $\sigma_a$ and $\sigma_b$ at that point for the given loading condition. We then plot the point of coordinates $\sigma_a$ and $\sigma_b$. If this point falls within a certain area, the component is safe; if it falls outside, the component will fail. The area used with the maximum-shearing-strength criterion is shown in Fig. 7.84 and the area used with the maximum-distortion-energy criterion in Fig. 7.85. We note that both areas depend upon the value of the yield strength $\sigma_Y$ of the material.
Fracture criteria for brittle materials under plane stress were developed in Sec. 7.8 in a similar fashion. The most commonly used is Mohr's criterion, which utilizes the results of various types of test available for a given material. The shaded area shown in Fig. 7.86 is used when the ultimate strengths $\sigma_{UT}$ and $\sigma_{UC}$ have been determined, respectively, from a tension and a compression test. Again, the principal stresses $\sigma_a$ and $\sigma_b$ are determined at a given point of the structural or machine component being investigated. If the corresponding point falls within the shaded area, the component is safe; if it falls outside, the component will rupture.

![Fig. 7.86](image)

Fracture criteria for brittle materials

In Sec. 7.9, we discussed the stresses in thin-walled pressure vessels and derived formulas relating the stresses in the walls of the vessels and the gage pressure $p$ in the fluid they contain. In the case of a cylindrical vessel of inside radius $r$ and thickness $t$ (Fig. 7.87), we obtained the following expressions for the hoop stress $\sigma_1$ and the longitudinal stress $\sigma_2$:

$$\sigma_1 = \frac{pr}{t}, \quad \sigma_2 = \frac{pr}{2t} \quad (7.30, 7.31)$$

We also found that the maximum shearing stress occurs out of the plane of stress and is

$$\tau_{\text{max}} = \sigma_2 = \frac{pr}{2t} \quad (7.34)$$

In the case of a spherical vessel of inside radius $r$ and thickness $t$ (Fig. 7.88), we found that the two principal stresses are equal:

$$\sigma_1 = \sigma_2 = \frac{pr}{2t} \quad (7.36)$$

Again, the maximum shearing stress occurs out of the plane of stress; it is

$$\tau_{\text{max}} = \frac{1}{2}\sigma_1 = \frac{pr}{4t} \quad (7.37)$$

![Fig. 7.87](image)

Cylindrical pressure vessels

![Fig. 7.88](image)

Spherical pressure vessels
Transformations of Stress and Strain

Transformation of plane strain

The last part of the chapter was devoted to the transformation of strain. In Secs. 7.10 and 7.11, we discussed the transformation of plane strain and introduced Mohr's circle for plane strain. The discussion was similar to the corresponding discussion of the transformation of stress, except that, where the shearing stress $\tau$ was used, we now used $\gamma$, that is, half the shearing strain. The formulas obtained for the transformation of strain under a rotation of axes through an angle $\theta$ were

$$
\varepsilon_x = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta + \gamma_{xy} \sin 2\theta \quad (7.44)
$$

$$
\varepsilon_y = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta - \gamma_{xy} \sin 2\theta \quad (7.45)
$$

$$
\gamma_{xy} = -(\varepsilon_x - \varepsilon_y) \sin 2\theta + \gamma_{xy} \cos 2\theta \quad (7.49)
$$

Using Mohr's circle for strain (Fig. 7.89), we also obtained the following relations defining the angle of rotation $\theta$, corresponding to the principal axes of strain and the values of the principal strains $\varepsilon_{\text{max}}$ and $\varepsilon_{\text{min}}$:

$$
\tan 2\theta = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} \quad (7.52)
$$

$$
\varepsilon_{\text{max}} = \varepsilon_{\text{ave}} + R \quad \text{and} \quad \varepsilon_{\text{min}} = \varepsilon_{\text{ave}} - R \quad (7.51)
$$

where

$$
\varepsilon_{\text{ave}} = \frac{\varepsilon_x + \varepsilon_y}{2} \quad \text{and} \quad R = \sqrt{\frac{(\varepsilon_x - \varepsilon_y)^2}{2} + \gamma_{xy}^2} \quad (7.50)
$$

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The maximum shearing strain for a rotation in the plane of strain was found to be

$$
\gamma_{\text{max (in plane)}} = 2R = \sqrt{(\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}^2} \quad (7.53)
$$

Section 7.12 was devoted to the three-dimensional analysis of strain, with application to the determination of the maximum shearing strain in the particular cases of plane strain and plane stress. In the case of plane stress, we also found that the principal strain $\varepsilon$, in a direction perpendicular to the plane of stress could be expressed as follows in terms of the “in-plane” principal strains $\varepsilon_a$ and $\varepsilon_b$:

$$
\varepsilon = -\frac{1}{1 - \nu}(\varepsilon_a + \varepsilon_b) \quad (7.59)
$$

Finally, we discussed in Sec. 7.13 the use of strain gages to measure the normal strain on the surface of a structural element or machine component. Considering a strain gage consisting of three gages aligned along lines forming respectively, angles $\theta_1$, $\theta_2$, and $\theta_3$ with the $x$ axis (Fig. 7.90), we wrote the following relations among the measurements $\varepsilon_1, \varepsilon_2, \varepsilon_3$ of the gages and the components $\varepsilon_x, \varepsilon_y, \varepsilon_{xy}$ characterizing the state of strain at that point:

$$
\varepsilon_1 = \varepsilon_x \cos^2 \theta_1 + \varepsilon_y \sin^2 \theta_1 + \gamma_{xy} \sin \theta_1 \cos \theta_1
$$

$$
\varepsilon_2 = \varepsilon_x \cos^2 \theta_2 + \varepsilon_y \sin^2 \theta_2 + \gamma_{xy} \sin \theta_2 \cos \theta_2 \quad (7.60)
$$

$$
\varepsilon_3 = \varepsilon_x \cos^2 \theta_3 + \varepsilon_y \sin^2 \theta_3 + \gamma_{xy} \sin \theta_3 \cos \theta_3
$$

These equations can be solved for $\varepsilon_x, \varepsilon_y$, and $\gamma_{xy}$ once $\varepsilon_1, \varepsilon_2$, and $\varepsilon_3$ have been determined.

Fig. 7.89

Mohr's circle for strain

Fig. 7.90

Strain gages. Strain rosette
**REVIEW PROBLEMS**

### 7.158
Two wooden members of $80 \times 120$-mm uniform rectangular cross section are joined by the simple glued scarf splice shown. Knowing that $\beta = 22^\circ$ and that the maximum allowable stresses in the joint are, respectively, 400 kPa in tension (perpendicular to the splice) and 600 kPa in shear (parallel to the splice), determine the largest centric load $P$ that can be applied.

![Fig. P7.158 and P7.159](image)

### 7.159
Two wooden members of $80 \times 120$-mm uniform rectangular cross section are joined by the simple glued scarf splice shown. Knowing that $\beta = 25^\circ$ and that centric loads of magnitude $P = 10$ kN are applied to the members as shown, determine (a) the in-plane shearing stress parallel to the splice, (b) the normal stress perpendicular to the splice.

![Fig. P7.160](image)

### 7.160
The centric force $P$ is applied to a short post as shown. Knowing that the stresses on plane $a-a$ are $\sigma = -15$ ksi and $\tau = 5$ ksi, determine (a) the angle $\beta$ that plane $a-a$ forms with the horizontal, (b) the maximum compressive stress in the post.

### 7.161
Determine the principal planes and the principal stresses for the state of plane stress resulting from the superposition of the two states of stress shown.

![Fig. P7.161](image)
7.162 For the state of stress shown, determine the maximum shearing stress when (a) \( \sigma_z = +24 \text{ MPa} \), (b) \( \sigma_z = -24 \text{ MPa} \), (c) \( \sigma_z = 0 \).

![Figure P7.162](http://www.opoosoft.com)

7.163 For the state of stress shown, determine the maximum shearing stress when (a) \( \tau_{yz} = 17.5 \text{ ksi} \), (b) \( \tau_{yz} = 8 \text{ ksi} \), (c) \( \tau_{yz} = 0 \).

![Figure P7.163](http://www.opoosoft.com)

7.164 The state of plane stress shown occurs in a machine component made of a steel with \( \sigma_y = 30 \text{ ksi} \). Using the maximum-distortion-energy criterion, determine whether yield will occur when (a) \( \tau_{xy} = 6 \text{ ksi} \), (b) \( \tau_{xy} = 12 \text{ ksi} \), (c) \( \tau_{xy} = 14 \text{ ksi} \). If yield does not occur, determine the corresponding factor of safety.

![Figure P7.164](http://www.opoosoft.com)
7.165 A torque of magnitude \( T = 12 \text{ kN} \cdot \text{m} \) is applied to the end of a tank containing compressed air under a pressure of 8 MPa. Knowing that the tank has a 180-mm inner diameter and a 12-mm wall thickness, determine the maximum normal stress and the maximum shearing stress in the tank.

7.166 The tank shown has a 180-mm inner diameter and a 12-mm wall thickness. Knowing that the tank contains compressed air under a pressure of 8 MPa, determine the magnitude \( T \) of the applied torque for which the maximum normal stress is 75 MPa.

7.167 The brass pipe \( AD \) is fitted with a jacket used to apply a hydrostatic pressure of 500 psi to portion \( BC \) of the pipe. Knowing that the pressure inside the pipe is 100 psi, determine the maximum normal stress in the pipe.

7.168 For the assembly of Prob. 7.167, determine the normal stress in the jacket (a) in a direction perpendicular to the longitudinal axis of the jacket, (b) in a direction parallel to that axis.

7.169 Determine the largest in-plane normal strain, knowing that the following strains have been obtained by the use of the rosette shown:

\[
\varepsilon_1 = -50 \times 10^{-6} \text{ in./in.} \quad \varepsilon_2 = +360 \times 10^{-6} \text{ in./in.} \quad \varepsilon_3 = +315 \times 10^{-6} \text{ in./in.}
\]
The following problems are to be solved with a computer.

7.C1 A state of plane stress is defined by the stress components $\sigma_x$, $\sigma_y$, and $\tau_{xy}$ associated with the element shown in Fig. P7.C1a. (a) Write a computer program that can be used to calculate the stress components $\sigma'_x$, $\sigma'_y$, and $\tau'_{xy}$ associated with the element after it has rotated through an angle $\theta$ about the $z$ axis (Fig. P7.C1b). (b) Use this program to solve Probs. 7.13 through 7.16.

![Fig. P7.C1](image)

7.C2 A state of plane stress is defined by the stress components $\sigma_x$, $\sigma_y$, and $\tau_{xy}$ associated with the element shown in Fig. P7.C1a. (a) Write a computer program that can be used to calculate the principal axes, the principal stresses, the maximum in-plane shearing stress, and the maximum shearing stress. (b) Use this program to solve Probs. 7.5, 7.9, 7.68, and 7.69.

7.C3 (a) Write a computer program that, for a given state of plane stress and a given yield strength of a ductile material, can be used to determine whether the material will yield. The program should use both the maximum shearing-strength criterion and the maximum-distortion-energy criterion. It should also print the values of the principal stresses and, if the material does not yield, calculate the factor of safety. (b) Use this program to solve Probs. 7.81, 7.82, and 7.164.

7.C4 (a) Write a computer program based on Mohr's fracture criterion for brittle materials that, for a given state of plane stress and given values of the ultimate strength of the material in tension and compression, can be used to determine whether rupture will occur. The program should also print the values of the principal stresses. (b) Use this program to solve Probs. 7.91 and 7.92 and to check the answers to Probs. 7.93 and 7.94.
7.C5 A state of plane strain is defined by the strain components $\varepsilon_x$, $\varepsilon_y$, and $\gamma_{xy}$ associated with the x and y axes. (a) Write a computer program that can be used to calculate the strain components $\varepsilon_x$, $\varepsilon_y$, and $\gamma_{xy}$ associated with the frame of reference $x'y'$ obtained by rotating the x and y axes through an angle $\theta$. (b) Use this program to solve Probs. 7.129 and 7.131.

![Fig. P7.C5](image)

7.C6 A state of strain is defined by the strain components $\varepsilon_x$, $\varepsilon_y$, and $\gamma_{xy}$ associated with the x and y axes. (a) Write a computer program that can be used to determine the orientation and magnitude of the principal strains, the maximum in-plane shearing strain, and the maximum shearing strain. (b) Use this program to solve Probs. 7.136 through 7.139.

7.C7 A state of plane strain is defined by the strain components $\varepsilon_x$, $\varepsilon_y$, and $\gamma_{xy}$ measured at a point. (a) Write a computer program that can be used to determine the orientation and magnitude of the principal strains, the maximum in-plane shearing strain, and the magnitude of the shearing strain. (b) Use this program to solve Probs. 7.140 through 7.143.

7.C8 A rosette consisting of three gages forming, respectively, angles of $\theta_1$, $\theta_2$, and $\theta_3$ with the x axis is attached to the free surface of a machine component made of a material with a given Poisson's ratio $\nu$. (a) Write a computer program that, for given readings $\varepsilon_1$, $\varepsilon_2$, and $\varepsilon_3$ of the gages, can be used to calculate the strain components associated with the x and y axes and to determine the orientation and magnitude of the three principal strains, the maximum in-plane shearing strain, and the maximum shearing strain. (b) Use this program to solve Probs. 7.144, 7.145, 7.146, and 7.169.
Due to gravity and wind load, the post supporting the sign shown is subjected simultaneously to compression, bending, and torsion. In this chapter you will learn to determine the stresses created by such combined loadings in structures and machine components.
CHAPTER 8

Principal Stresses under a Given Loading

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Chapter 8 Principal Stresses under a Given Loading

8.1 Introduction
8.2 Principal Stresses in a Beam
8.3 Design of Transmission Shafts
8.4 Stresses under Combined Loadings

*8.1 INTRODUCTION

In the first part of this chapter, you will apply to the design of beams and shafts the knowledge that you acquired in Chap. 7 on the transformation of stresses. In the second part of the chapter, you will learn how to determine the principal stresses in structural members and machine elements under given loading conditions.

In Chap. 5 you learned to calculate the maximum normal stress \( \sigma_m \) occurring in a beam under a transverse loading (Fig. 8.1a) and check whether this value exceeded the allowable stress \( \sigma_{all} \) for the given material. If it did, the design of the beam was not acceptable. While the danger for a brittle material is actually to fail in tension, the danger for a ductile material is to fail in shear (Fig. 8.1b). The fact that \( \sigma_m > \sigma_{all} \) indicates that \( |M|_{max} \) is too large for the cross section selected, but does not provide any information on the actual mechanism of failure. Similarly, the fact that \( \tau_m > \tau_{all} \) simply indicates that \( |V|_{max} \) is too large for the cross section selected. While the danger for a ductile material is actually to fail in shear (Fig. 8.2a), the danger for a brittle material is to fail in tension under the principal stresses (Fig. 8.2b). The distribution of the principal stresses in a beam will be discussed in Sec. 8.2.

Depending upon the shape of the cross section of the beam and the value of the shear \( V \) in the critical section where \( |M| = |M|_{max} \), it may happen that the largest value of the normal stress will not occur at the top or bottom of the section, but at some other point within the section. As you will see in Sec. 8.2, a combination of large values of \( \sigma_x \) and \( \tau_{xy} \) near the junction of the web and the flanges of a W-beam or an S-beam can result in a value of the principal stress \( \sigma_{max} \) (Fig. 8.3) that is larger than the value of \( \sigma_m \) on the surface of the beam.

Section 8.3 will be devoted to the design of transmission shafts subjected to transverse loads as well as to torques. The effect of both the normal stresses due to bending and the shearing stresses due to torsion will be taken into account.

In Sec. 8.4 you will learn to determine the stresses at a given point \( K \) of a body of arbitrary shape subjected to a combined loading. First, you will reduce the given loading to forces and couples in the section containing \( K \). Next, you will calculate the normal and shearing stresses at \( K \). Finally, using one of the methods for the transformation of stresses that you learned in Chap. 7, you will determine the principal planes, principal stresses, and maximum shearing stress at \( K \).
**8.2 PRINCIPAL STRESSES IN A BEAM**

Consider a prismatic beam $AB$ subjected to some arbitrary transverse loading (Fig. 8.4). We denote by $V$ and $M$, respectively, the shear and bending moment in a section through a given point $C$. We recall from Chaps. 5 and 6 that, within the elastic limit, the stresses exerted on a small element with faces perpendicular, respectively, to the $x$ and $y$ axes reduce to the normal stresses $\sigma_m = Mc/I$ if the element is at the free surface of the beam, and to the shearing stresses $\tau_m = VQ/I$ if the element is at the neutral surface (Fig. 8.5).

At any other point of the cross section, an element of material is subjected simultaneously to the normal stresses

$$\sigma_x = -\frac{My}{I} \quad (8.1)$$

where $y$ is the distance from the neutral surface and $I$ the centroidal moment of inertia of the section, and to the shearing stresses

$$\tau_{xy} = -\frac{VQ}{It} \quad (8.2)$$

where $Q$ is the first moment about the neutral axis of the portion of the cross-sectional area located above the point where the stresses are computed, and $t$ the width of the cross section at that point. Using either of the methods of analysis presented in Chap. 7, we can obtain the principal stresses at any point of the cross section (Fig. 8.6).

The following question now arises: Can the maximum normal stress $\sigma_{\max}$ at some point within the cross section be larger than the value of $\sigma_m = Mc/I$ computed at the surface of the beam? If it can, then the determination of the largest normal stress in the beam will involve a great deal more than the computation of $|M|_{\max}$ and the use of Eq. (8.1). We can obtain an answer to this question by investigating the distribution of the principal stresses in a narrow
Principal Stresses under a Given Loading

A rectangular cantilever beam subjected to a concentrated load \( P \) at its free end (Fig. 8.7). We recall from Sec. 6.5 that the normal and shearing stresses at a distance \( x \) from the load \( P \) and a distance \( y \) above the neutral surface are given, respectively, by Eq. (6.13) and Eq. (6.12). Since the moment of inertia of the cross section is

\[
I = \frac{bh^3}{12} = \frac{(bh)(2c)^2}{12} = \frac{Ac^2}{3}
\]

where \( A \) is the cross-sectional area and \( c \) the half-depth of the beam, we write

\[
\sigma_x = \frac{Pxy}{I} = \frac{Pxy}{\frac{1}{3}Ac^2} = \frac{3}{A} \frac{Pxy}{Ac^2} \tag{8.3}
\]

and

\[
\tau_{xy} = \frac{3}{2} \frac{P}{A} \left( 1 - \frac{y^2}{c^2} \right) \tag{8.4}
\]

Using the method of Sec. 7.3 or Sec. 7.4, the value of \( \sigma_{\text{max}} \) can be determined at any point of the beam. Figure 8.8 shows the results of the computation of the ratios \( \sigma_{\text{max}}/\sigma_{\text{m}} \) and \( \sigma_{\text{min}}/\sigma_{\text{m}} \) in two sections of the beam, corresponding respectively to \( x = 2c \) and \( x = 8c \). In

<table>
<thead>
<tr>
<th>( x = 2c )</th>
<th>( \sigma_{\text{min}}/\sigma_{\text{m}} )</th>
<th>( \sigma_{\text{max}}/\sigma_{\text{m}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0</td>
<td>1.000</td>
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<td>0.8</td>
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<td>0.810</td>
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<tr>
<td>0.6</td>
<td>-0.040</td>
<td>0.640</td>
</tr>
<tr>
<td>0.4</td>
<td>-0.090</td>
<td>0.490</td>
</tr>
<tr>
<td>0.2</td>
<td>-0.160</td>
<td>0.360</td>
</tr>
<tr>
<td>0</td>
<td>-0.230</td>
<td>0.250</td>
</tr>
<tr>
<td>-0.2</td>
<td>-0.360</td>
<td>0.160</td>
</tr>
<tr>
<td>-0.4</td>
<td>-0.490</td>
<td>0.090</td>
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<tr>
<td>-0.6</td>
<td>-0.640</td>
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<tr>
<td>-0.8</td>
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<td>0.010</td>
</tr>
<tr>
<td>-1.0</td>
<td>-1.000</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ x = 8c \]

<table>
<thead>
<tr>
<th>( \sigma_{\text{min}}/\sigma_{\text{m}} )</th>
<th>( \sigma_{\text{max}}/\sigma_{\text{m}} )</th>
</tr>
</thead>
<tbody>
<tr>
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<tr>
<td>-0.063</td>
<td>0.063</td>
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<tr>
<td>-0.217</td>
<td>0.017</td>
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<td>-0.407</td>
<td>0.007</td>
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<tr>
<td>-0.603</td>
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<tr>
<td>-0.801</td>
<td>0.001</td>
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<tr>
<td>-1.000</td>
<td>0</td>
</tr>
</tbody>
</table>

Fig. 8.8 Distribution of principal stresses in two transverse sections of a rectangular cantilever beam supporting a single concentrated load.
each section, these ratios have been determined at 11 different points, and the orientation of the principal axes has been indicated at each point.†

It is clear that $\sigma_{\text{max}}$ does not exceed $\sigma_m$ in either of the two sections considered in Fig. 8.8 and that, if it does exceed $\sigma_m$ elsewhere, it will be in sections close to the load $P$, where $\sigma_m$ is small compared to $\tau_m$.‡ But, for sections close to the load $P$, Saint-Venant's principle does not apply, Eqs. (8.3) and (8.4) cease to be valid, except in the very unlikely case of a load distributed parabolically over the end section (cf. Sec. 6.5), and more advanced methods of analysis taking into account the effect of stress concentrations should be used. We thus conclude that, for beams of rectangular cross section, and within the scope of the theory presented in this text, the maximum normal stress can be obtained from Eq. (8.1).

In Fig. 8.8 the directions of the principal axes were determined at 11 points in each of the two sections considered. If this analysis were extended to a larger number of sections and a larger number of points in each section, it would be possible to draw two orthogonal systems of curves on the side of the beam (Fig. 8.9). One system would consist of curves tangent to the principal axes corresponding to $\sigma_{\text{max}}$ and the other of curves tangent to the principal axes corresponding to $\sigma_{\text{min}}$. The curves obtained in this manner are known as the stress trajectories. A trajectory of the first group (solid lines) defines at each of its points the direction of the largest tensile stress, while a trajectory of the second group (dashed lines) defines the direction of the largest compressive stress.§

The conclusion we have reached for beams of rectangular cross section, that the maximum normal stress in the beam can be obtained from Eq. (8.1), remains valid for many beams of nonrectangular cross section. However, when the width of the cross section varies in such a way that large shearing stresses $\tau_{xy}$ will occur at points close to the surface of the beam, where $\sigma_x$ is also large, a value of the principal stress $\sigma_{\text{max}}$ larger than $\sigma_m$ may result at such points. One should be particularly aware of this possibility when selecting W-beams or S-beams, and calculate the principal stress $\sigma_{\text{max}}$ at the junctions $b$ and $d$ of the web with the flanges of the beam (Fig. 8.10). This is done by determining $\sigma_x$ and $\tau_{xy}$ at that point from Eqs. (8.1) and (8.2), respectively, and using either of the methods of analysis of Chap. 7 to obtain $\sigma_{\text{max}}$ (see Sample Prob. 8.1). An alternative procedure, used in design to select an acceptable section, consists of using for $\tau_{xy}$ the maximum value of the shearing stress in the section, $\tau_{\text{max}} = V/A_{\text{web}}$ given by Eq. (6.11) of Sec. 6.4. This leads to a slightly larger, and thus conservative, value of the principal stress $\sigma_{\text{max}}$ at the junction of the web with the flanges of the beam (see Sample Prob. 8.2).

†See Prob. 8.C2, which refers to a program that can be written to obtain the results shown in Fig. 8.8.

‡As will be verified in Prob. 8.C2, $\sigma_{\text{max}}$ exceeds $\sigma_m$ if $x \leq 0.544c$.

§A brittle material, such as concrete, will fail in tension along planes that are perpendicular to the tensile-stress trajectories. Thus, to be effective, steel reinforcing bars should be placed so that they intersect these planes. On the other hand, stiffeners attached to the web of a plate girder will be effective in preventing buckling only if they intersect planes perpendicular to the compressive-stress trajectories.
8.3 DESIGN OF TRANSMISSION SHAFTS

When we discussed the design of transmission shafts in Sec. 3.7, we considered only the stresses due to the torques exerted on the shafts. However, if the power is transferred to and from the shaft by means of gears or sprocket wheels (Fig. 8.11a), the forces exerted on the gear teeth or sprockets are equivalent to force-couple systems applied at the centers of the corresponding cross sections (Fig. 8.11b). This means that the shaft is subjected to a transverse loading, as well as to a torsional loading.

The shearing stresses produced in the shaft by the transverse loads are usually much smaller than those produced by the torques and will be neglected in this analysis.† The normal stresses due to the transverse loads, however, may be quite large and, as you will see presently, their contribution to the maximum shearing stress \( \tau_{\text{max}} \) should be taken into account.

†For an application where the shearing stresses produced by the transverse loads must be considered, see Probs. 8.21 and 8.22.
Consider the cross section of the shaft at some point C. We represent the torque \( T \) and the bending couples \( M_y \) and \( M_z \) acting, respectively, in a horizontal and a vertical plane by the couple vectors shown (Fig. 8.12a). Since any diameter of the section is a principal axis of inertia for the section, we can replace \( M_y \) and \( M_z \) by their resultant \( M \) (Fig. 8.12b) in order to compute the normal stresses exerted on the section. We thus find that \( \sigma_z \) is maximum at the end of the diameter perpendicular to the vector representing \( M \) (Fig. 8.13). Recalling that the values of the normal stresses at that point are, respectively, \( \sigma_m = Mc/I \) and zero, while the shearing stress is \( \tau_m = Tc/J \), we plot the corresponding points \( X \) and \( Y \) on a Mohr-circle diagram (Fig. 8.14) and determine the value of the maximum shearing stress:

\[
\tau_{\text{max}} = R = \sqrt{\left(\frac{\sigma_m}{2}\right)^2 + (\tau_m)^2} = \sqrt{\left(\frac{Mc}{2I}\right)^2 + \left(\frac{Tc}{J}\right)^2} 
\]

Recalling that, for a circular or annular cross section, \( 2I = J \), we write

\[
\tau_{\text{max}} = \frac{c}{J} \sqrt{M^2 + T^2} \quad (8.5)
\]

It follows that the minimum allowable value of the area \( J/c \) for the cross section of the shaft is

\[
\frac{J}{c} = \frac{\left(\sqrt{M^2 + T^2}\right)_{\text{max}}}{\tau_{\text{all}}} \quad (8.6)
\]

where the numerator in the right-hand member of the expression obtained represents the maximum value of \( \sqrt{M^2 + T^2} \) in the shaft, and \( \tau_{\text{all}} \) the allowable shearing stress. Expressing the bending moment \( M \) in terms of its components in the two coordinate planes, we can also write

\[
\frac{J}{c} = \frac{\left(\sqrt{M_y^2 + M_z^2 + T^2}\right)_{\text{max}}}{\tau_{\text{all}}} \quad (8.7)
\]

Equations (8.6) and (8.7) can be used to design both solid and hollow circular shafts and should be compared with Eq. (3.22) of Sec. 3.7, which was obtained under the assumption of a torsional loading only.

The determination of the maximum value of \( \sqrt{M_y^2 + M_z^2 + T^2} \) will be facilitated if the bending-moment diagrams corresponding to \( M_y \) and \( M_z \) are drawn, as well as a third diagram representing the values of \( T \) along the shaft (see Sample Prob. 8.3).
SAMPLE PROBLEM 8.1

A 160-kN force is applied as shown at the end of a W200 × 52 rolled-steel beam. Neglecting the effect of fillets and of stress concentrations, determine whether the normal stresses in the beam satisfy a design specification that they be equal to or less than 150 MPa at section A-A'.

SOLUTION

Shear and Bending Moment. At section A-A', we have

\[ M_A = (160 \, \text{kN})(0.375 \, \text{m}) = 60 \, \text{kN} \cdot \text{m} \]
\[ V_A = 160 \, \text{kN} \]

Normal Stresses on Transverse Plane. Referring to the table of Properties of Rolled-Steel Shapes in Appendix C, we obtain the data shown and then determine the stresses \( \sigma_a \) and \( \sigma_b \).

At point a:

\[ \sigma_a = \frac{M_A}{S} = \frac{60 \, \text{kN} \cdot \text{m}}{511 \times 10^{-6} \, \text{m}^3} = 117.4 \, \text{MPa} \]

At point b:

\[ \sigma_b = \sigma_a \frac{y_b}{c} = (117.4 \, \text{MPa}) \frac{90.4 \, \text{mm}}{103 \, \text{mm}} = 103.0 \, \text{MPa} \]

We note that all normal stresses on the transverse plane are less than 150 MPa.

Shearing Stresses on Transverse Plane

At point a:

\[ Q = 0 \quad \tau_a = 0 \]

At point b:

\[ Q = (206 \times 12.6)(96.7) = 251.0 \times 10^3 \, \text{mm}^3 = 251.0 \times 10^{-6} \, \text{m}^3 \]
\[ \tau_b = \frac{V_A Q}{I t} = \frac{(160 \, \text{kN})(251.0 \times 10^{-6} \, \text{m}^3)}{52.9 \times 10^{-6} \, \text{m}^3 \times 0.00787 \, \text{m}} = 96.5 \, \text{MPa} \]

Principal Stress at Point b. The state of stress at point b consists of the normal stress \( \sigma_b = 103.0 \, \text{MPa} \) and the shearing stress \( \tau_b = 96.5 \, \text{MPa} \). We draw Mohr’s circle and find

\[ \sigma_{\text{max}} = \frac{1}{2} \sigma_b + R = \frac{1}{2} \sigma_b + \sqrt{\left(\frac{1}{2} \sigma_b\right)^2 + \tau_b^2} \]
\[ = \frac{103.0}{2} + \sqrt{\left(\frac{103.0}{2}\right)^2 + (96.5)^2} \]
\[ = 160.9 \, \text{MPa} \]

The specification, \( \sigma_{\text{max}} \leq 150 \, \text{MPa} \), is not satisfied.

Comment. For this beam and loading, the principal stress at point b is 36% larger than the normal stress at point a. For \( L \approx 881 \, \text{mm} \), the maximum normal stress would occur at point a.
SAMPLE PROBLEM 8.2

The overhanging beam AB supports a uniformly distributed load of 3.2 kips/ft and a concentrated load of 20 kips at C. Knowing that for the grade of steel to be used \( \sigma_{u} = 24 \) ksi and \( \tau_{u} = 14.5 \) ksi, select the wide-flange shape that should be used.

SOLUTION

Reactions at A and D. We draw the free-body diagram of the beam. From the equilibrium equations \( \Sigma M_{D} = 0 \) and \( \Sigma M_{A} = 0 \) we find the values of \( R_{A} \) and \( R_{D} \) shown in the diagram.

Shear and Bending-Moment Diagrams. Using the methods of Secs. 5.2 and 5.3, we draw the diagrams and observe that

\[
|M|_{\text{max}} = 239.4 \text{ kip} \cdot \text{ft} = 2873 \text{ kip} \cdot \text{in.} \quad |V|_{\text{max}} = 43 \text{ kips}
\]

Section Modulus. For \( |M|_{\text{max}} = 2873 \text{ kip} \cdot \text{in.} \) and \( \sigma_{u} = 24 \) ksi, the minimum acceptable section modulus of the rolled-steel shape is

\[
S_{\text{min}} = \frac{|M|_{\text{max}}}{\sigma_{u}} = \frac{2873 \text{ kip} \cdot \text{in.}}{24 \text{ ksi}} = 119.7 \text{ in}^{3}
\]

Selection of Wide-Flange Shape. From the table of Properties of Rolled-Steel Shapes in Appendix C, we compile a list of the lightest shapes of a given depth that have a section modulus larger than \( S_{\text{min}} \).

<table>
<thead>
<tr>
<th>Shape</th>
<th>( S ) (in(^{3}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>W24 × 68</td>
<td>154</td>
</tr>
<tr>
<td>W21 × 62</td>
<td>127</td>
</tr>
<tr>
<td>W18 × 76</td>
<td>146</td>
</tr>
<tr>
<td>W16 × 77</td>
<td>134</td>
</tr>
<tr>
<td>W14 × 82</td>
<td>123</td>
</tr>
<tr>
<td>W12 × 96</td>
<td>131</td>
</tr>
</tbody>
</table>

We now select the lightest shape available, namely \( W21 \times 62 \).

Shearing Stress. Since we are designing the beam, we will conservatively assume that the maximum shear is uniformly distributed over the web area of a \( W21 \times 62 \). We write

\[
\tau_{w} = \frac{V_{\text{max}}}{A_{\text{web}}} = \frac{43 \text{ kips}}{8.40 \text{ in}^{2}} = 5.12 \text{ ksi} < 14.5 \text{ ksi} \quad \text{(OK)}
\]

Principal Stress at Point b. We check that the maximum principal stress at point b in the critical section where \( M \) is maximum does not exceed \( \sigma_{u} = 24 \) ksi. We write

\[
\sigma_{a} = \frac{M_{\text{max}}}{S} = \frac{2873 \text{ kip} \cdot \text{in.}}{127 \text{ in}^{3}} = 22.6 \text{ ksi}
\]

\[
\sigma_{b} = \frac{y_{b}}{c} \cdot \frac{\sigma_{a}}{c} = (22.6 \text{ ksi}) \frac{9.88 \text{ in.}}{10.50 \text{ in.}} = 21.3 \text{ ksi}
\]

Conservatively, \( \tau_{b} = \frac{V}{A_{\text{web}}} = \frac{12.2 \text{ kips}}{8.40 \text{ in}^{2}} = 1.45 \text{ ksi} \)

We draw Mohr’s circle and find

\[
\sigma_{\text{max}} = \frac{1}{2} \sigma_{b} + R = \frac{21.3 \text{ ksi}}{2} + \sqrt{\left( \frac{21.3 \text{ ksi}}{2} \right)^{2} + (1.45 \text{ ksi})^{2}}
\]

\[
\sigma_{\text{max}} = 21.4 \text{ ksi} \leq 24 \text{ ksi} \quad \text{(OK)}
\]
SAMPLE PROBLEM 8.3

The solid shaft AB rotates at 480 rpm and transmits 30 kW from the motor M to machine tools connected to gears G and H. 20 kW is taken off at gear G and 10 kW at gear H. Knowing that \( \tau_{all} = 50 \text{ MPa} \), determine the smallest permissible diameter for shaft AB.

**SOLUTION**

**Torques Exerted on Gears.** Observing that \( f = 480 \text{ rpm} = 8 \text{ Hz} \), we determine the torque exerted on gear E:

\[
T_E = \frac{P}{2\pi f} = \frac{30 \text{ kW}}{2\pi (8 \text{ Hz})} = 597 \text{ N} \cdot \text{m}
\]

The corresponding tangential force acting on the gear is

\[
F_E = \frac{T_E}{r_E} = \frac{597 \text{ N} \cdot \text{m}}{0.16 \text{ m}} = 3.73 \text{ kN}
\]

A similar analysis of gears C and D yields

\[
T_C = \frac{20 \text{ kW}}{2\pi (8 \text{ Hz})} = 398 \text{ N} \cdot \text{m} \quad F_C = 6.63 \text{ kN}
\]

\[
T_D = \frac{10 \text{ kW}}{2\pi (8 \text{ Hz})} = 199 \text{ N} \cdot \text{m} \quad F_D = 2.49 \text{ kN}
\]

We now replace the forces on the gears by equivalent force-couple systems.

**Bending-Moment and Torque Diagrams**

**Critical Transverse Section.** By computing \( \sqrt{M_y^2 + M_z^2 + T^2} \) at all potentially critical sections, we find that its maximum value occurs just to the right of D:

\[
\sqrt{M_y^2 + M_z^2 + T^2_{max}} = \sqrt{(1160)^2 + (373)^2 + (597)^2} = 1357 \text{ N} \cdot \text{m}
\]

**Diameter of Shaft.** For \( \tau_{all} = 50 \text{ MPa} \), Eq. (7.32) yields

\[
\frac{J}{c} = \frac{\sqrt{M_y^2 + M_z^2 + T^2_{max}}}{\tau_{all}} = \frac{1357 \text{ N} \cdot \text{m}}{50 \text{ MPa}} = 27.14 \times 10^{-6} \text{ m}^3
\]

For a solid circular shaft of radius \( c \), we have

\[
\frac{J}{c^3} = \frac{\pi}{2} c^3 = 27.14 \times 10^{-6} \quad c = 0.02585 \text{ m} = 25.85 \text{ mm}
\]

Diameter = 2c = 51.7 mm

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8.1 A W10 × 39 rolled-steel beam supports a load P as shown. Knowing that \( P = 45 \text{ kips} \), \( a = 10 \text{ in.} \), and \( \sigma_{\text{all}} = 18 \text{ ksi} \), determine (a) the maximum value of the normal stress \( \sigma_m \) in the beam, (b) the maximum value of the principal stress \( \sigma_{\text{max}} \) at the junction of the flange and web, (c) whether the specified shape is acceptable as far as these two stresses are concerned.

8.2 Solve Prob. 8.1, assuming that \( P = 22.5 \text{ kips} \) and \( a = 20 \text{ in.} \).

8.3 An overhanging W920 × 449 rolled-steel beam supports a load P as shown. Knowing that \( P = 700 \text{ kN} \), \( a = 2.5 \text{ m} \), and \( \sigma_{\text{all}} = 100 \text{ MPa} \), determine (a) the maximum value of the normal stress \( \sigma_m \) in the beam, (b) the maximum value of the principal stress \( \sigma_{\text{max}} \) at the junction of the flange and web, (c) whether the specified shape is acceptable as far as these two stresses are concerned.

8.4 Solve Prob. 8.3, assuming that \( P = 850 \text{ kN} \) and \( a = 2.0 \text{ m} \).

8.5 and 8.6 (a) Knowing that \( \sigma_{\text{all}} = 24 \text{ ksi} \) and \( \tau_{\text{all}} = 14.5 \text{ ksi} \), select the most economical wide-flange shape that should be used to support the loading shown. (b) Determine the values to be expected for \( \sigma_m \), \( \tau_m \), and the principal stress \( \sigma_{\text{max}} \) at the junction of a flange and the web of the selected beam.

8.7 and 8.8 (a) Knowing that \( \sigma_{\text{all}} = 160 \text{ MPa} \) and \( \tau_{\text{all}} = 100 \text{ MPa} \), select the most economical metric wide-flange shape that should be used to support the loading shown. (b) Determine the values to be expected for \( \sigma_m \), \( \tau_m \), and the principal stress \( \sigma_{\text{max}} \) at the junction of a flange and the web of the selected beam.
8.9 through 8.14 Each of the following problems refers to a rolled-steel shape selected in a problem of Chap. 5 to support a given loading at a minimal cost while satisfying the requirement $\sigma_{m} \leq \sigma_{all}$. For the selected design, determine (a) the actual value $\sigma_{m}$ in the beam, (b) the maximum value of the principal stress $\sigma_{max}$ at the junction of a flange and the web.

8.9 Loading of Prob. 5.73 and selected W530 \times 66 shape.
8.10 Loading of Prob. 5.74 and selected W530 \times 92 shape.
8.11 Loading of Prob. 5.77 and selected S15 \times 42.9 shape.
8.12 Loading of Prob. 5.78 and selected S12 \times 31.8 shape.
8.13 Loading of Prob. 5.75 and selected S460 \times 81.4 shape.
8.14 Loading of Prob. 5.76 and selected S510 \times 98.2 shape.

8.15 The vertical force $P_{1}$ and the horizontal force $P_{2}$ are applied as shown to disks welded to the solid shaft $AD$. Knowing that the diameter of the shaft is 1.75 in. and that $\tau_{all} = 8 \text{ ksi}$, determine the largest permissible magnitude of the force $P_{2}$.

![Fig. P8.15](image1)

8.16 The two 500-lb forces are vertical and the force $P$ is parallel to the $z$ axis. Knowing that $\tau_{all} = 8 \text{ ksi}$, determine the smallest permissible diameter of the solid shaft $AE$.

![Fig. P8.16](image2)

8.17 For the gear-and-shaft system and loading of Prob. 8.16, determine the smallest permissible diameter of shaft $AE$, knowing that the shaft is hollow and has an inner diameter that is $\frac{1}{2}$ the outer diameter.

8.18 The 4-kN force is parallel to the $x$ axis, and the force $Q$ is parallel to the $z$ axis. The shaft $AD$ is hollow. Knowing that the inner diameter is half the outer diameter and that $\tau_{all} = 60 \text{ MPa}$, determine the smallest permissible outer diameter of the shaft.
8.19 Neglecting the effect of fillets and of stress concentrations, determine the smallest permissible diameters of the solid rods BC and CD. Use $\tau_{all} = 60$ MPa.

![Fig. P8.19 and P8.20]

8.20 Knowing that rods BC and CD are of diameter 24 mm and 36 mm, respectively, determine the maximum shearing stress in each rod. Neglect the effect of fillets and of stress concentrations.

8.21 It was stated in Sec. 8.3 that the shearing stresses produced in a shaft by the transverse loads are usually much smaller than those produced by the torques. In the preceding problems their effect was ignored and it was assumed that the maximum shearing stress in a given section occurred at point H (Fig. P8.21a) and was equal to the expression obtained in Eq. (8.5), namely,

$$\tau_H = \frac{c}{J} \sqrt{M^2 + T^2}$$

Show that the maximum shearing stress at point K (Fig. P8.21b), where the effect of the shear V is greatest, can be expressed as

$$\tau_K = \frac{c}{J} \sqrt{(M \cos \beta)^2 + \left(\frac{2}{3} V + T\right)^2}$$

where $\beta$ is the angle between the vectors V and M. It is clear that the effect of the shear V cannot be ignored when $\tau_K \approx \tau_H$. (Hint: Only the component of M along V contributes to the shearing stress at K.)

8.22 Assuming that the magnitudes of the forces applied to disks A and C of Prob. 8.15 are, respectively, $P_1 = 1080$ lb and $P_2 = 810$ lb, and using the expressions given in Prob. 8.21, determine the values of $\tau_H$ and $\tau_K$ in a section (a) just to the left of B, (b) just to the left of C.

8.23 The solid shafts ABC and DEF and the gears shown are used to transmit 20 hp from the motor M to a machine tool connected to shaft DEF. Knowing that the motor rotates at 240 rpm and that $\tau_{all} = 7.5$ ksi, determine the smallest permissible diameter of (a) shaft ABC, (b) shaft DEF.

8.24 Solve Prob. 8.23, assuming that the motor rotates at 360 rpm.
8.25 The solid shaft $AB$ rotates at 360 rpm and transmits 20 kW from the motor $M$ to machine tools connected to gears $E$ and $F$. Knowing that $\tau_{all} = 45$ MPa and assuming that 10 kW is taken off at each gear, determine the smallest permissible diameter of shaft $AB$.

8.26 Solve Prob. 8.25 assuming that the entire 20 kW is taken off at gear $E$.

8.27 The solid shaft $ABC$ and the gears shown are used to transmit 10 kW from the motor $M$ to a machine tool connected to gear $D$. Knowing that the motor rotates at 240 rpm and that $\tau_{all} = 60$ MPa, determine the smallest permissible diameter of shaft $ABC$.

8.28 Assuming that shaft $ABC$ of Prob. 8.27 is hollow and has an outer diameter of 50 mm, determine the largest permissible inner diameter of the shaft.
8.29 The solid shaft $AE$ rotates at 600 rpm and transmits 60 hp from the motor $M$ to machine tools connected to gears $G$ and $H$. Knowing that $\tau_{ij} = 8$ ksi and that 40 hp is taken off at gear $G$ and 20 hp is taken off at gear $H$, determine the smallest permissible diameter of shaft $AE$.

---

8.30 Solve Prob. 8.29, assuming that 30 hp is taken off at gear $G$ and 30 hp is taken off at gear $H$.

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8.4 STRESSES UNDER COMBINED LOADINGS

In Chaps. 1 and 2 you learned to determine the stresses caused by a centric axial load. In Chap. 3, you analyzed the distribution of stresses in a cylindrical member subjected to a twisting couple. In Chap. 4, you determined the stresses caused by bending couples and, in Chaps. 5 and 6, the stresses produced by transverse loads. As you will see presently, you can combine the knowledge you have acquired to determine the stresses in slender structural members or machine components under fairly general loading conditions.

Consider, for example, the bent member $ABDE$ of circular cross section that is subjected to several forces (Fig. 8.15). In order to determine the stresses produced at points $H$ or $K$ by the given loads, we first pass a section through these points and determine the force-couple system at the centroid $C$ of the section that is required to maintain the equilibrium of portion $ABC$.† This system represents the internal forces in the section and, in general, consists of three

†The force-couple system at $C$ can also be defined as equivalent to the forces acting on the portion of the member located to the right of the section (see Example 8.01).
Principal Stresses under a Given Loading

The force $P$ is a centric axial force that produces normal stresses in the section. The couple vectors $M_y$ and $M_z$ cause the member to bend and also produce normal stresses in the section. They have therefore been grouped with the force $P$ in part (a) of Fig. 8.17 and the sums $s_x$ of the normal stresses they produce at points $H$ and $K$ have been shown in part (a) of Fig. 8.18. These stresses can be determined as shown in Sec. 4.14.

On the other hand, the twisting couple $T$ and the shearing forces $V_y$ and $V_z$ produce shearing stresses in the section. The sums $t_{xy}$ and $t_{xz}$ of the components of the shearing stresses they produce at points $H$ and $K$ have been shown in part (b) of Fig. 8.18 and can be determined as indicated in Secs. 3.4 and 6.3.† The normal and shearing stresses shown in parts (a) and (b) of Fig. 8.18 can now be combined and displayed at points $H$ and $K$ on the surface of the member (Fig. 8.19).

The principal stresses and the orientation of the principal planes at points $H$ and $K$ can be determined from the values of $s_x$, $t_{xy}$, and $t_{xz}$ at each of these points by one of the methods presented in Chap. 7 (Fig. 8.20). The values of the maximum shearing stress at each of these points and the corresponding planes can be found in a similar way.

The results obtained in this section are valid only to the extent that the conditions of applicability of the superposition principle (Sec. 2.12) and of Saint-Venant's principle (Sec. 2.17) are met. This means that the stresses involved must not exceed the proportional limit of the material, that the deformations due to one of the loadings must not affect the determination of the stresses due to the others, and that the section used in your analysis must not be too close to the points of application of the given forces. It is clear from the first of these requirements that the method presented here cannot be applied to plastic deformations.

†Note that your present knowledge allows you to determine the effect of the twisting couple $T$ only in the cases of circular shafts, of members with a rectangular cross section (Sec. 3.12), or of thin-walled hollow members (Sec. 3.13).
Two forces \( \mathbf{P}_1 \) and \( \mathbf{P}_2 \) of magnitude \( P_1 = 15 \text{ kN} \) and \( P_2 = 18 \text{ kN} \), are applied as shown to the end \( A \) of bar \( AB \), which is welded to a cylindrical member \( BD \) of radius \( c = 20 \text{ mm} \) (Fig. 8.21). Knowing that the distance from \( A \) to the axis of member \( BD \) is \( a = 50 \text{ mm} \) and assuming that all stresses remain below the proportional limit of the material, determine (a) the normal and shearing stresses at point \( K \) of the transverse section of member \( BD \) located at a distance \( b = 60 \text{ mm} \) from end \( B \), (b) the principal axes and principal stresses at \( K \), (c) the maximum shearing stress at \( K \).

**Internal Forces in Given Section.** We first replace the forces \( \mathbf{P}_1 \) and \( \mathbf{P}_2 \) by an equivalent system of forces and couples applied at the center \( C \) of the section containing point \( K \) (Fig. 8.22). This system, which represents the internal forces in the section, consists of the following forces and couples:

1. A centric axial force \( \mathbf{F} \) equal to the force \( \mathbf{P}_1 \), of magnitude 
   \[ F = P_1 = 15 \text{ kN} \]
2. A shearing force \( \mathbf{V} \) equal to the force \( \mathbf{P}_2 \), of magnitude 
   \[ V = P_2 = 18 \text{ kN} \]
3. A twisting couple \( \mathbf{T} \) of torque \( T \) equal to the moment of \( \mathbf{P}_2 \) about the axis of member \( BD \):
   \[ T = P_2 a = (18 \text{ kN})(50 \text{ mm}) = 900 \text{ N} \cdot \text{m} \]
4. A bending couple \( \mathbf{M}_y \), of moment \( M_y \) equal to the moment of \( \mathbf{P}_1 \) about a vertical axis through \( C \):
   \[ M_y = P_1 a = (15 \text{ kN})(50 \text{ mm}) = 750 \text{ N} \cdot \text{m} \]
5. A bending couple \( \mathbf{M}_z \), of moment \( M_z \), equal to the moment of \( \mathbf{P}_2 \) about a transverse, horizontal axis through \( C \):
   \[ M_z = P_2 b = (18 \text{ kN})(60 \text{ mm}) = 1080 \text{ N} \cdot \text{m} \]

The results obtained are shown in Fig. 8.23.

**a. Normal and Shearing Stresses at Point \( K \).** Each of the forces and couples shown in Fig. 8.23 can produce a normal or shearing stress at point \( K \). Our purpose is to compute separately each of these stresses, and then to add the normal stresses and add the shearing stresses. But we must first determine the geometric properties of the section.

**Geometric Properties of the Section** We have

\[
A = \pi c^2 = \pi (0.020 \text{ m})^2 = 1.257 \times 10^{-3} \text{ m}^2
\]

\[
I_y = I_z = \frac{1}{12} \pi c^4 = \frac{1}{12} \pi (0.020 \text{ m})^4 = 125.7 \times 10^{-9} \text{ m}^4
\]

\[
J_C = \frac{1}{2} \pi c^4 = \frac{1}{2} \pi (0.020 \text{ m})^4 = 251.3 \times 10^{-9} \text{ m}^4
\]

We also determine the first moment \( Q \) and the width \( t \) of the area of the cross section located above the \( z \) axis. Recalling that \( y = 4c/3\pi \) for a semicircle of radius \( c \), we have

\[
Q = A y = \left( \frac{1}{2} \pi c^2 \right) \left( \frac{4c}{3\pi} \right) = \frac{2}{3} c^3 = \frac{2}{3} (0.020 \text{ m})^3
\]

and

\[ t = 2c = 2(0.020 \text{ m}) = 0.040 \text{ m} \]

**Normal Stresses.** We observe that normal stresses are produced at \( K \) by the centric force \( \mathbf{F} \) and the bending couple \( \mathbf{M}_y \), but that the couple \( \mathbf{M}_z \)
Fig. 8.26

Fig. 8.27

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does not produce any stress at \( K \), since \( K \) is located on the neutral axis corresponding to that couple. Determining each sign from Fig. 8.23, we write

\[
\sigma_x = - \frac{F}{A} + \frac{M_y C}{I_y} = -11.9 \text{ MPa} + \frac{(750 \text{ N} \cdot \text{m})(0.020 \text{ m})}{125.7 \times 10^{-3} \text{ m}^4}
\]
\[
= -11.9 \text{ MPa} + 119.3 \text{ MPa}
\]
\[
\sigma_x = +107.4 \text{ MPa}
\]

**Shearing Stresses.** These consist of the shearing stress \( \tau_{xy}/V \) due to the vertical shear \( V \) and of the shearing stress \( \tau_{xy}/V_{\text{twist}} \) caused by the torque \( T \). Recalling the values obtained for \( Q, t, I_1, \) and \( J_C \), we write

\[
\tau_{xy}/V = \frac{1}{I_1} \left( 18 \times 10^3 \text{ N} \cdot (5.33 \times 10^{-6} \text{ m}^3) \right)
\]
\[
= +19.1 \text{ MPa}
\]

\[
\tau_{xy}/V_{\text{twist}} = \frac{-T}{J} = \frac{-900 \text{ N} \cdot \text{m}}{251.3 \times 10^{-3} \text{ m}^4}
\]
\[
= -71.6 \text{ MPa}
\]

Adding these two expressions, we obtain \( \tau_{xy} \) at point \( K \).

\[
\tau_{xy} = (\tau_{xy})_V + (\tau_{xy})_{V_{\text{twist}}} = +19.1 \text{ MPa} - 71.6 \text{ MPa}
\]
\[
\tau_{xy} = -52.5 \text{ MPa}
\]

In Fig. 8.24, the normal stress \( \sigma_x \) and the shearing stresses and \( \tau_{xy} \) have been shown acting on a square element located at \( K \) on the surface of the cylindrical member. Note that shearing stresses acting on the longitudinal sides of the element have been included.

**b. Principal Planes and Principal Stresses at Point \( K \).** We can use either of the two methods of Chap. 7 to determine the principal planes and principal stresses. Following Mohr’s circle, we plot point \( \alpha \) of coordinates \( \sigma_x = +107.4 \text{ MPa} \) and \( \tau_{xy} = +52.5 \text{ MPa} \) and point \( \gamma \) of coordinates \( \sigma_y = 0 \) and \( \tau_{xy} = -52.5 \text{ MPa} \) and draw the circle of diameter \( XY \) (Fig. 8.25). Observing that

\[
OC = CD = \frac{1}{2} (107.4) = 53.7 \text{ MPa}
\]
\[
DX = 52.5 \text{ MPa}
\]

we determine the orientation of the principal planes:

\[
\tan 2\theta_p = \frac{DX}{CD} = \frac{52.5}{53.7} = 0.97765
\]
\[
2\theta_p = 44.4^\circ \downarrow
\]
\[
\theta_p = 22.2^\circ \downarrow
\]

We now determine the radius of the circle,

\[
R = \sqrt{(53.7)^2 + (52.5)^2} = 75.1 \text{ MPa}
\]

and the principal stresses,

\[
\sigma_{\text{max}} = OC + R = 53.7 + 75.1 = 128.8 \text{ MPa}
\]
\[
\sigma_{\text{min}} = OC - R = 53.7 - 75.1 = -21.4 \text{ MPa}
\]

The results obtained are shown in Fig. 8.26.

**c. Maximum Shearing Stress at Point \( K \).** This stress corresponds to points \( E \) and \( F \) in Fig. 8.25. We have

\[
\tau_{\text{max}} = CE = R = 75.1 \text{ MPa}
\]

Observing that \( 2\theta_p = 90^\circ - 2\theta_p = 90^\circ - 44.4^\circ = 45.6^\circ \), we conclude that the planes of maximum shearing stress form an angle \( \theta_p = 22.8^\circ \) with the horizontal. The corresponding element is shown in Fig. 8.27. Note that the normal stresses acting on this element are represented by \( OC \) in Fig. 8.25 and are thus equal to +53.7 MPa.
SAMPLE PROBLEM 8.4

A horizontal 500-lb force acts at point D of crankshaft AB which is held in static equilibrium by a twisting couple T and by reactions at A and B. Knowing that the bearings are self-aligning and exert no couples on the shaft, determine the normal and shearing stresses at points H, J, K, and L located at the ends of the vertical and horizontal diameters of a transverse section located 2.5 in. to the left of bearing B.

SOLUTION

Free Body. Entire Crankshaft. \( A = B = 250 \text{ lb} \)

\[ + \sum \tau = 0: \quad -250 \text{ lb(1.8 in.)} + T = 0 \quad T = 900 \text{ lb \cdot in.} \]

Internal Forces in Transverse Section. We replace the reaction B and the twisting couple T by an equivalent force-couple system at the center C of the transverse section containing H, J, K, and L.

\[ V = B = 250 \text{ lb} \quad T = 900 \text{ lb \cdot in.} \]

\[ M_y = (250 \text{ lb})(2.5 \text{ in.}) = 625 \text{ lb \cdot in.} \]

The geometric properties of the 0.9-in.-diameter section are

\[ A = \pi(0.45 \text{ in.})^2 = 0.636 \text{ in}^2 \quad I = \frac{1}{4}\pi(0.45 \text{ in.})^4 = 32.2 \times 10^{-3} \text{ in}^4 \]

\[ J = \frac{1}{4}\pi(0.45 \text{ in.})^4 = 64.4 \times 10^{-3} \text{ in}^4 \]

Stresses Produced by Twisting Couple T. Using Eq. (3.8), we determine the shearing stresses at points H, J, K, and L and show them in Fig. (a).

\[ \tau = \frac{Tc}{J} = \frac{(900 \text{ lb \cdot in.})(0.45 \text{ in.})}{64.4 \times 10^{-3} \text{ in}^4} = 6290 \text{ psi} \]

Stresses Produced by Shearing Force V. The shearing force V produces no shearing stresses at points J and L. At points H and K we first compute \( Q \) for a semicircle about a vertical diameter and then determine the shearing stress produced by the shear force V = 250 lb. These stresses are shown in Fig. (b).

\[ Q = \frac{1}{2} \pi r^2 \left( \frac{4c}{3\pi} \right) = \frac{2}{3} c^3 = \frac{2}{3} (0.45 \text{ in.})^3 = 60.7 \times 10^{-3} \text{ in}^3 \]

\[ \tau = \frac{VQ}{It} = \frac{(250 \text{ lb})(60.7 \times 10^{-3} \text{ in}^3)}{(32.2 \times 10^{-3} \text{ in}^4)(0.9 \text{ in.})} = 524 \text{ psi} \]

Stresses Produced by the Bending Couple \( M_y \). Since the bending couple \( M_y \) acts in a horizontal plane, it produces no stresses at H and K. Using Eq. (4.15), we determine the normal stresses at points J and L and show them in Fig. (c).

\[ \sigma = \frac{|M_y|c}{I} = \frac{(625 \text{ lb \cdot in.})(0.45 \text{ in.})}{32.2 \times 10^{-3} \text{ in}^4} = 8730 \text{ psi} \]

Summary. We add the stresses shown and obtain the total normal and shearing stresses at points H, J, K, and L.
SAMPLE PROBLEM 8.5

Three forces are applied as shown at points A, B, and D of a short steel post. Knowing that the horizontal cross section of the post is a 40 \times 140-mm rectangle, determine the principal stresses, principal planes and maximum shearing stress at point H.

**INTERNAL FORCES IN SECTION EFG.** We replace the three applied forces by an equivalent force-couple system at the center C of the rectangular section EFG. We have

\[ V_x = -30 \text{kN} \quad P = 50 \text{kN} \quad V_z = -75 \text{kN} \]

\[ M_x = (50 \text{kN})(0.130 \text{m}) - (75 \text{kN})(0.200 \text{m}) = -8.5 \text{kN} \cdot \text{m} \]

\[ M_y = 0 \quad M_z = (30 \text{kN})(0.100 \text{m}) = 3 \text{kN} \cdot \text{m} \]

We note that there is no twisting couple about the y axis. The geometric properties of the rectangular section are

\[ A = (0.040 \text{m})(0.140 \text{m}) = 5.6 \times 10^{-3} \text{m}^2 \]

\[ I_x = \frac{1}{12}(0.040 \text{m})(0.140 \text{m})^3 = 9.15 \times 10^{-6} \text{m}^4 \]

\[ I_y = \frac{1}{12}(0.140 \text{m})(0.040 \text{m})^3 = 0.747 \times 10^{-6} \text{m}^4 \]

**NORMAL STRESS AT H.** We note that normal stresses \( \sigma \) are produced by the centric force \( P \) and by the bending couples \( M_x \) and \( M_z \). We determine the sign of each stress by carefully examining the sketch of the force-couple system at C.

\[ \sigma_y = \frac{P}{A} + \frac{|M_x| a}{I_x} - \frac{|M_z| b}{I_y} \]

\[ = \frac{50 \text{kN}}{5.6 \times 10^{-3} \text{m}^2} + \frac{(3 \text{kN} \cdot \text{m})(0.020 \text{m})}{9.15 \times 10^{-6} \text{m}^4} - \frac{(8.5 \text{kN} \cdot \text{m})(0.025 \text{m})}{0.747 \times 10^{-6} \text{m}^4} \]

\[ \sigma_y = 8.93 \text{MPa} + 80.3 \text{MPa} - 23.2 \text{MPa} \]

\[ \sigma_y = 66.0 \text{MPa} \]

**SHEARING STRESS AT H.** Considering first the shearing force \( V_z \), we note that \( Q = 0 \) with respect to the z axis, since \( H \) is on the edge of the cross section. Thus \( V_z \) produces no shearing stress at \( H \). The shearing force \( V_z \) does produce a shearing stress at \( H \) and we write

\[ Q = A_t \gamma_z = \frac{(0.040 \text{m})(0.045 \text{m})(0.0475 \text{m})}{6 \text{m}^4} = 85.5 \times 10^{-6} \text{m}^3 \]

\[ \gamma_z = \frac{V_z Q}{I_t} = \frac{(75 \text{kN})(85.5 \times 10^{-6} \text{m}^3)}{(9.15 \times 10^{-6} \text{m}^4)(0.040 \text{m})} \]

\[ \tau_{yz} = 17.52 \text{MPa} \]

**PRINCIPAL STRESSES, PRINCIPAL PLANES, AND MAXIMUM SHEARING STRESS AT H.** We draw Mohr’s circle for the stresses at point H.

\[ \tan 2\theta_p = \frac{17.52}{33.0} \quad 2\theta_p = 27.96^\circ \quad \theta_p = 13.98^\circ \]

\[ R = \sqrt{(33.0)^2 + (17.52)^2} = 37.4 \text{MPa} \]

\[ \tau_{\text{max}} = 37.4 \text{MPa} \]

\[ \sigma_{\text{max}} = OA = OC + R = 33.0 + 37.4 \]

\[ \sigma_{\text{max}} = 70.4 \text{MPa} \]

\[ \sigma_{\text{min}} = OB = OC - R = 33.0 - 37.4 \]

\[ \sigma_{\text{min}} = -7.4 \text{MPa} \]
8.31 A 6-kip force is applied to the machine element \( AB \) as shown. Knowing that the uniform thickness of the element is 0.8 in., determine the normal and shearing stresses at (a) point \( a \), (b) point \( b \), (c) point \( c \).

![Fig. P8.31 and P8.32](image)

8.32 A 6-kip force is applied to the machine element \( AB \) as shown. Knowing that the uniform thickness of the element is 0.8 in., determine the normal and shearing stresses at (a) point \( d \), (b) point \( e \), (c) point \( f \).

8.33 For the bracket and loading shown, determine the normal and shearing stresses at (a) point \( a \), (b) point \( b \).

8.34 through 8.36 Member \( AB \) has a uniform rectangular cross section of 10 × 24 mm. For the loading shown, determine the normal and shearing stresses at (a) point \( H \), (b) point \( K \).

![Fig. P8.34](image)

![Fig. P8.35](image)

![Fig. P8.36](image)
Principal Stresses under a Given Loading

8.37 Several forces are applied to the pipe assembly shown. Knowing that the pipe has an inner and outer diameters equal to 1.61 and 1.90 in., respectively, determine the normal and shearing stresses at (a) point $H$, (b) point $K$.

![Diagram of a pipe assembly with forces applied](Fig_P8.37)

8.38 The steel pile $AB$ has a 100-mm outer diameter and an 8-mm wall thickness. Knowing that the tension in the cable is 40 kN, determine the normal and shearing stresses at point $H$.

![Diagram of a steel pile with forces applied](Fig_P8.38)

8.39 The billboard shown weighs 8000 lb and is supported by a structural tube that has a 15-in. outer diameter and a 0.5-in. wall thickness. At a time when the resultant of the wind pressure is 3 kips, located at the center $C$ of the billboard, determine the normal and shearing stresses at point $H$.

![Diagram of a billboard with forces and stresses](Fig_P8.39)

8.40 A thin strap is wrapped around a solid rod of radius $c = 20$ mm as shown. Knowing that $l = 100$ mm and $F = 5$ kN, determine the normal and shearing stresses at (a) point $H$, (b) point $K$.

![Diagram of a strap wrapped around a rod](Fig_P8.40)
8.41 A vertical force \( P \) of magnitude 60 lb is applied to the crank at point \( A \). Knowing that the shaft \( BDE \) has a diameter of 0.75 in., determine the principal stresses and the maximum shearing stress at point \( H \) located at the top of the shaft, 2 in. to the right of support \( D \).

8.42 A 13-kN force is applied as shown to the 60-mm-diameter cast-iron post \( ABD \). At point \( H \), determine (a) the principal stresses and principal planes, (b) the maximum shearing stress.

8.43 A 10-kN force and a 1.4-kN-m couple are applied at the top of the 65-mm diameter brass post shown. Determine the principal stresses and maximum shearing stress at (a) point \( H \), (b) point \( K \).

8.44 Forces are applied at points \( A \) and \( B \) of the solid cast-iron bracket shown. Knowing that the bracket has a diameter of 0.8 in., determine the principal stresses and the maximum shearing stress at (a) point \( H \), (b) point \( K \).

8.45 Three forces are applied to the bar shown. Determine the normal and shearing stresses at (a) point \( a \), (b) point \( b \), (c) point \( c \).

8.46 Solve Prob. 8.45, assuming that \( h = 12 \) in.
8.47 Three forces are applied to the bar shown. Determine the normal and shearing stresses at (a) point a, (b) point b, (c) point c.

8.48 Solve Prob. 8.47, assuming that the 750-N force is directed vertically upward.

8.49 For the post and loading shown, determine the principal stresses, principal planes, and maximum shearing stress at point H.

8.50 For the post and loading shown, determine the principal stresses, principal planes, and maximum shearing stress at point K.

8.51 Two forces are applied to the small post BD as shown. Knowing that the vertical portion of the post has a cross section of 1.5 x 2.4 in., determine the principal stresses, principal planes, and maximum shearing stress at point H.
8.52 Solve Prob. 8.51, assuming that the magnitude of the 6000-lb force is reduced to 1500 lb.

8.53 Three steel plates, each 13 mm thick, are welded together to form a cantilever beam. For the loading shown, determine the normal and shearing stresses at points $a$ and $b$.

![Diagram of three steel plates welded together](image1)

Fig. P8.53 and P8.54

8.54 Three steel plates, each 13 mm thick, are welded together to form a cantilever beam. For the loading shown, determine the normal and shearing stresses at points $d$ and $e$.

8.55 Two forces are applied to a W8 × 28 rolled-steel beam as shown. Determine the principal stresses and maximum shearing stress at point $a$.

![Diagram of W8 × 28 rolled-steel beam](image2)

Fig. P8.55 and P8.56

8.56 Two forces are applied to a W8 × 28 rolled-steel beam as shown. Determine the principal stresses and maximum shearing stress at point $b$. 
8.57 Two forces $P_1$ and $P_2$ are applied as shown in directions perpendicular to the longitudinal axis of a W310 × 60 beam. Knowing that $P_1 = 25 \text{ kN}$ and $P_2 = 24 \text{ kN}$, determine the principal stresses and the maximum shearing stress at point $a$.

![Fig. P8.57 and P8.58](image)

8.58 Two forces $P_1$ and $P_2$ are applied as shown in directions perpendicular to the longitudinal axis of a W310 × 60 beam. Knowing that $P_1 = 25 \text{ kN}$ and $P_2 = 24 \text{ kN}$, determine the principal stresses and the maximum shearing stress at point $b$.

8.59 A vertical force $P$ is applied at the center of the free end of cantilever beam AB. When the beam is installed with the web vertical ($\beta = 0$) and with its longitudinal axis AB horizontal, determine the magnitude of the force $P$ for which the normal stress at point $a$ is $+120 \text{ MPa}$. 

(b) Solve part a, assuming that the beam is installed with $\beta = 3^\circ$.

![Fig. P8.59](image)

8.60 A force $P$ is applied to a cantilever beam by means of a cable attached to a bolt located at the center of the free end of the beam. Knowing that $P$ acts in a direction perpendicular to the longitudinal axis of the beam, determine (a) the normal stress at point $a$ in terms of $P$, $b$, $h$, $l$, and $\beta$, (b) the values of $\beta$ for which the normal stress at $a$ is zero.

![Fig. P8.60](image)
**8.61** A 5-kN force \( P \) is applied to a wire that is wrapped around bar \( AB \) as shown. Knowing that the cross section of the bar is a square of side \( d = 40 \text{ mm} \), determine the principal stresses and the maximum shearing stress at point \( a \).

![Fig. P8.61](image)

**8.62** Knowing that the structural tube shown has a uniform wall thickness of 0.3 in., determine the principal stresses, principal planes, and maximum shearing stress at (a) point \( H \), (b) point \( K \).

![Fig. P8.62](image)

**8.63** The structural tube shown has a uniform wall thickness of 0.3 in. Knowing that the 15-kip load is applied 0.15 in. above the base of the tube, determine the shearing stress at (a) point \( a \), (b) point \( b \).

![Fig. P8.63](image)

**8.64** For the tube and loading of Prob. 8.63, determine the principal stresses and the maximum shearing stress at point \( b \).
This chapter was devoted to the determination of the principal stresses in beams, transmission shafts, and bodies of arbitrary shape subjected to combined loadings.

We first recalled in Sec. 8.2 the two fundamental relations derived in Chaps. 5 and 6 for the normal stress $\sigma_x$ and the shearing stress $\tau_{xy}$ at any given point of a cross section of a prismatic beam,

$$\sigma_x = -\frac{My}{I}, \quad \tau_{xy} = -\frac{VQ}{It} \quad (8.1, 8.2)$$

where $V =$ shear in the section
$M =$ bending moment in the section
$y =$ distance of the point from the neutral surface
$I =$ centroidal moment of inertia of the cross section
$Q =$ first moment about the neutral axis of the portion of the cross section located above the given point
$t =$ width of the cross section at the given point

Using one of the methods presented in Chap. 7 for the transformation of stresses, we were able to obtain the principal planes and principal stresses at the given point (Fig. 8.28).

We investigated the distribution of the principal stresses in a narrow rectangular cantilever beam subjected to a concentrated load $P$ at its free end and found that in any given transverse section—except close to the point of application of the load—the maximum principal stress $\sigma_{\text{max}}$ did not exceed the maximum normal stress $\sigma_m$ occurring at the surface of the beam.

While this conclusion remains valid for many beams of non-rectangular cross section, it may not hold for W-beams or S-beams, where $\sigma_{\text{max}}$ at the junctions $b$ and $d$ of the web with the flanges of the beam (Fig. 8.29) may exceed the value of $\sigma_m$ occurring at points $a$ and $e$. Therefore, the design of a rolled-steel beam should include the computation of the maximum principal stress at these points. (See Sample Probs. 8.1 and 8.2.)
In Sec. 8.3, we considered the design of transmission shafts subjected to transverse loads as well as to torques. Taking into account the effect of both the normal stresses due to the bending moment $M$ and the shearing stresses due to the torque $T$ in any given transverse section of a cylindrical shaft (either solid or hollow), we found that the minimum allowable value of the ratio $J/c$ for the cross section was

$$\frac{J}{c} = \left(\frac{\sqrt{M^2 + T^2}}{\tau_{all}}\right)_{\text{max}} \quad (8.6)$$

In preceding chapters, you learned to determine the stresses in prismatic members caused by axial loadings (Chaps. 1 and 2), torsion (Chap. 3), bending (Chap. 4), and transverse loadings (Chaps. 5 and 6). In the second part of this chapter (Sec. 8.4), we combined this knowledge to determine stresses under more general loading conditions.

For instance, to determine the stresses at point $H$ or $K$ of the bent member shown in Fig. 8.30, we passed a section through these points and replaced the applied loads by an equivalent force-couple system at the centroid $C$ of the section (Fig. 8.31). The normal and shearing stresses produced at $H$ or $K$ by each of the forces and couples applied at $C$ were determined and then combined to obtain the resulting normal stress $\sigma_x$ and the resulting shearing stresses $\tau_{xy}$ and $\tau_{xz}$ at $H$ or $K$. Finally, the principal stresses, the orientation of the principal planes, and the maximum shearing stress at point $H$ or $K$ were determined by one of the methods presented in Chap. 7 from the values obtained for $\sigma_x$, $\tau_{xy}$, and $\tau_{xz}$.
8.65 (a) Knowing that $\sigma_{\text{all}} = 24$ ksi and $\tau_{\text{all}} = 14.5$ ksi, select the most economical wide-flange shape that should be used to support the loading shown. (b) Determine the values to be expected for $\sigma_{\text{fm}}$, $\tau_{\text{fm}}$, and the principal stress $\sigma_{\text{max}}$ at the junction of a flange and the web of the selected beam.

![Diagram](http://www.opoosoft.com)

8.66 Determine the smallest allowable diameter of the solid shaft $ABCD$, knowing that $\tau_{\text{all}} = 60$ MPa and that the radius of disk $B$ is $r = 80$ mm.

8.67 Using the notation of Sec. 8.3 and neglecting the effect of shearing stresses caused by transverse loads, show that the maximum normal stress in a circular shaft can be expressed as follows:

$$\sigma_{\text{max}} = \frac{C}{J} \left[ (M_y^2 + M_z^2) + (M_y^2 + M_z^2 + T^2) \right]_{\text{max}}$$

8.68 The solid shaft $AB$ rotates at 450 rpm and transmits 20 kW from the motor $M$ to machine tools connected to gears $F$ and $G$. Knowing that $\tau_{\text{all}} = 55$ MPa and assuming that 8 kW is taken off at gear $F$ and 12 kW is taken off at gear $G$, determine the smallest permissible diameter of shaft $AB$.

![Diagram](http://www.opoosoft.com)
8.69 Two 1.2-kip forces are applied to an L-shaped machine element AB as shown. Determine the normal and shearing stresses at (a) point a, (b) point b, (c) point c.

Fig. P8.69

8.70 Two forces are applied to the pipe AB as shown. Knowing that the pipe has inner and outer diameters equal to 35 and 42 mm, respectively, determine the normal and shearing stresses at (a) point a, (b) point b.

8.71 A close-coiled spring is made of a circular wire of radius \( r \) that is formed into a helix of radius \( R \). Determine the maximum shearing stress produced by the two equal and opposite forces \( P \) and \( P' \). (Hint: First determine the shear \( V \) and the torque \( T \) in a transverse cross section.)

8.72 Three forces are applied to a 4-in.-diameter plate that is attached to the solid 1.8-in. diameter shaft AB. At point H, determine (a) the principal stresses and principal planes, (b) the maximum shearing stress.
8.73 Knowing that the bracket $AB$ has a uniform thickness of $\frac{5}{8}$ in., determine (a) the principal planes and principal stresses at point $K$, (b) the maximum shearing stress at point $K$.

8.74 Three forces are applied to the machine component $ABD$ as shown. Knowing that the cross section containing point $H$ is a $20 \times 40$-mm rectangle, determine the principal stresses and the maximum shearing stress at point $H$.

8.75 Knowing that the structural tube shown has a uniform wall thickness of 0.25 in., determine the normal and shearing stresses at the three points indicated.

8.76 The cantilever beam $AB$ will be installed so that the 60-mm side forms an angle $\beta$ between 0 and $90^\circ$ with the vertical. Knowing that the 600-N vertical force is applied at the center of the free end of the beam, determine the normal stress at point $a$ when (a) $\beta = 0$, (b) $\beta = 90^\circ$. (c) Also, determine the value of $\beta$ for which the normal stress at point $a$ is a maximum and the corresponding value of that stress.
The following problems are designed to be solved with a computer.

**8.C1** Let us assume that the shear $V$ and the bending moment $M$ have been determined in a given section of a rolled-steel beam. Write a computer program to calculate in that section, from the data available in Appendix C, (a) the maximum normal stress $s_{\text{m}}$, (b) the principal stress $s_{\text{max}}$ at the junction of a flange and the web. Use this program to solve parts $a$ and $b$ of the following problems:

1. Prob. 8.1 (Use $V = 45$ kips and $M = 450$ kip \cdot in.)
2. Prob. 8.2 (Use $V = 22.5$ kips and $M = 450$ kip \cdot in.)
3. Prob. 8.3 (Use $V = 700$ kN and $M = 1750$ kN \cdot m)
4. Prob. 8.4 (Use $V = 850$ kN and $M = 1700$ kN \cdot m)

**8.C2** A cantilever beam $AB$ with a rectangular cross section of width $b$ and depth $2c$ supports a single concentrated load $P$ at its end $A$. Write a computer program to calculate, for any values of $x/c$ and $y/c$, (a) the ratios $s_{\text{max}}/s_{\text{m}}$ and $s_{\text{max}}/s_{\text{min}}$, where $s_{\text{max}}$ and $s_{\text{min}}$ are the principal stresses at point $K(x, y)$ and $s_{\text{m}}$ the maximum normal stress in the same transverse section, (b) the angle $\theta$ that the principal planes at $K$ form with a transverse and a horizontal plane through $K$. Use this program to check the values shown in Fig. 8.8 and to verify that $s_{\text{max}}$ exceeds $s_{\text{m}}$ if $x \geq 0.544c$, as indicated in the second footnote on page 517.

**8.C3** Disks $D_1, D_2, \ldots, D_n$ are attached as shown in Fig. 8.C3 to the solid shaft $AB$ of length $L$, uniform diameter $d$, and allowable shearing stress $\tau_{\text{all}}$. Forces $P_1, P_2, \ldots, P_n$ of known magnitude (except for one of them) are applied to the disks, either at the top or bottom of its vertical diameter, or at the left or right end of its horizontal diameter. Denoting by $r_i$ the radius of disk $D_i$ and by $c_i$ its distance from the support at $A$, write a computer program to calculate (a) the magnitude of the unknown force $P_i$, (b) the smallest permissible value of the diameter $d$ of shaft $AB$. Use this program to solve Prob. 8.18.
8.C4 The solid shaft $AB$ of length $L$, uniform diameter $d$, and allowable shearing stress $\tau_{\text{all}}$ rotates at a given speed expressed in rpm (Fig. 8.C4). Gears $G_1$, $G_2$, . . . , $G_n$ are attached to the shaft and each of these gears meshes with another gear (not shown), either at the top or bottom of its vertical diameter, or at the left or right end of its horizontal diameter. One of these gears is connected to a motor and the rest of them to various machine tools. Denoting by $r_i$ the radius of disk $G_i$, by $c_i$ its distance from the support at $A$, and by $P_i$ the power transmitted to that gear (+ sign) or taken off that gear (− sign), write a computer program to calculate the smallest permissible value of the diameter $d$ of shaft $AB$. Use this program to solve Probs. 8.27 and 8.68.

8.C5 Write a computer program that can be used to calculate the normal and shearing stresses at points with given coordinates $y$ and $z$ located on the surface of a machine part having a rectangular cross section. The internal forces are known to be equivalent to the force-couple system shown. Write the program so that the loads and dimensions can be expressed in either SI or U.S. customary units. Use this program to solve (a) Prob. 8.45b, (b) Prob. 8.47a.
**8.C6** Member AB has a rectangular cross section of 10 × 24 mm. For the loading shown, write a computer program that can be used to determine the normal and shearing stresses at points H and K for values of d from 0 to 120 mm, using 15-mm increments. Use this program to solve Prob. 8.35.

![Fig. P8.C6](image)

**8.C7** The structural tube shown has a uniform wall thickness of 0.3 in. A 9-kip force is applied at a bar (not shown) that is welded to the end of the tube. Write a computer program that can be used to determine, for any given value of c, the principal stresses, principal planes, and maximum shearing stress at point H for values of d from 3 in. to 3 in., using one-inch increments. Use this program to solve Prob. 8.62a.

![Fig. P8.C7](image)
The photo shows a multiple-girder bridge during construction. The design of the steel girders is based on both strength considerations and deflection evaluations.
CHAPTER 9

Deflection of Beams

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Chapter 9 Deflection of Beams

**9.1 INTRODUCTION**

In the preceding chapter we learned to design beams for strength. In this chapter we will be concerned with another aspect in the design of beams, namely, the determination of the deflection. Of particular interest is the determination of the maximum deflection of a beam under a given loading, since the design specifications of a beam will generally include a maximum allowable value for its deflection. Also of interest is that a knowledge of the deflections is required to analyze indeterminate beams. These are beams in which the number of reactions at the supports exceeds the number of equilibrium equations available to determine these unknowns.

We saw in Sec. 4.4 that a prismatic beam subjected to pure bending is bent into an arc of circle and that, within the elastic range, the curvature of the neutral surface can be expressed as

\[ \frac{1}{\rho} = \frac{M}{EI} \]  

(4.21)

where \( M \) is the bending moment, \( E \) the modulus of elasticity, and \( I \) the moment of inertia of the cross section about its neutral axis.

When a beam is subjected to a transverse loading, Eq. (4.21) remains valid for any given transverse section, provided that Saint-Venant's principle applies. However, both the bending moment and the curvature of the neutral surface will vary from section to section. Denoting by \( x \) the distance of the section from the left end of the beam, we write

\[ \frac{1}{\rho} = \frac{M(x)}{EI} \]

(9.1)

The knowledge of the curvature at various points of the beam will enable us to draw some general conclusions regarding the deformation of the beam under loading (Sec. 9.2).

To determine the slope and deflection of the beam at any given point, we first derive the following second-order linear differential equation, which governs the elastic curve characterizing the shape of the deformed beam (Sec. 9.3):

\[ \frac{d^2y}{dx^2} = \frac{M(x)}{EI} \]

If the bending moment can be represented for all values of \( x \) by a single function \( M(x) \), as in the case of the beams and loadings shown in Fig. 9.1, the slope \( \theta = dy/dx \) and the deflection \( y \) at any point of the beam may be obtained through two successive integrations. The two constants of integration introduced in the process will be determined from the boundary conditions indicated in the figure.

However, if different analytical functions are required to represent the bending moment in various portions of the beam, different differential equations will also be required, leading to
different functions defining the elastic curve in the various portions of the beam. In the case of the beam and loading of Fig. 9.2, for example, two differential equations are required, one for the portion of beam AD and the other for the portion DB. The first equation yields the functions $u_1$ and $y_1$, and the second the functions $u_2$ and $y_2$. Altogether, four constants of integration must be determined; two will be obtained by writing that the deflection is zero at A and B, and the other two by expressing that the portions of beam AD and DB have the same slope and the same deflection at D.

You will observe in Sec. 9.4 that in the case of a beam supporting a distributed load $w(x)$, the elastic curve can be obtained directly from $w(x)$ through four successive integrations. The constants introduced in this process will be determined from the boundary values of $V$, $M$, $\theta$, and $y$.

In Sec. 9.5, we will discuss statically indeterminate beams where the reactions at the supports involve four or more unknowns. The three equilibrium equations must be supplemented with equations obtained from the boundary conditions imposed by the supports.

The method described earlier for the determination of the elastic curve when several functions are required to represent the bending moment $M$ can be quite laborious, since it requires matching slopes and deflections at every transition point. You will see in Sec. 9.6 that the use of singularity functions (previously discussed in Sec. 5.5) considerably simplifies the determination of $\theta$ and $y$ at any point of the beam.

The next part of the chapter (Secs. 9.7 and 9.8) is devoted to the method of superposition, which consists of determining separately, and then adding, the slope and deflection caused by the various loads applied to a beam. This procedure can be facilitated by the use of the table in Appendix D, which gives the slopes and deflections of beams for various loadings and types of support.

In Sec. 9.9, certain geometric properties of the elastic curve will be used to determine the deflection and slope of a beam at a given point. Instead of expressing the bending moment as a function $M(x)$ and integrating this function analytically, the diagram representing the variation of $M/EI$ over the length of the beam will be drawn and two moment-area theorems will be derived. The first moment-area theorem will enable us to calculate the angle between the tangents to the beam at two points; the second moment-area theorem will be used to calculate the vertical distance from a point on the beam to a tangent at a second point.

The moment-area theorems will be used in Sec. 9.10 to determine the slope and deflection at selected points of cantilever beams and beams with symmetric loadings. In Sec. 9.11 you will find that in many cases the areas and moments of areas defined by the $M/EI$ diagram may be more easily determined if you draw the bending-moment diagram by parts. As you study the moment-area method, you will observe that this method is particularly effective in the case of beams of variable cross section.
Deflection of Beams

Beams with unsymmetric loadings and overhanging beams will be considered in Sec. 9.12. Since for an unsymmetric loading the maximum deflection does not occur at the center of a beam, you will learn in Sec. 9.13 how to locate the point where the tangent is horizontal in order to determine the maximum deflection. Section 9.14 will be devoted to the solution of problems involving statically indeterminate beams.

9.2 DEFORMATION OF A BEAM UNDER TRANSVERSE LOADING

At the beginning of this chapter, we recalled Eq. (4.21) of Sec. 4.4, which relates the curvature of the neutral surface and the bending moment in a beam in pure bending. We pointed out that this equation remains valid for any given transverse section of a beam subject to a transverse loading, provided that Saint-Venant’s principle applies. However, both the bending moment and the curvature of the neutral surface will vary from section to section. Denoting by \( x \) the distance of the section from the left end of the beam, we write

\[
\frac{1}{\rho} = \frac{M(x)}{EI} \tag{9.1}
\]

Consider, for example, a cantilever beam \( AB \) of length \( L \) subject to a concentrated load \( P \) at its free end \( A \) (Fig. 9.3a). We have \( M(x) = 2Px \) and, substituting into (9.1),

\[
\frac{1}{\rho} = -\frac{Px}{EI}
\]

which shows that the curvature of the neutral surface varies linearly with \( x \), from zero at \( A \), where \( \rho_A \) itself is infinite, to \(-PL/EI\) at \( B \), where \( |\rho_B| = EI/PL \) (Fig. 9.3b).

Consider now the overhanging beam \( AD \) of Fig. 9.4a that supports two concentrated loads as shown. From the free-body diagram of the beam (Fig. 9.4b), we find that the reactions at the supports are \( R_A = 1 \) kN and \( R_C = 5 \) kN, respectively, and draw the corresponding bending-moment diagram (Fig. 9.5a). We note from the diagram that \( M \), and thus the curvature of the beam, are both zero at each end of the beam, and also at a point \( E \) located at \( x = 4 \) m. Between \( A \) and \( E \) the bending moment is positive and the beam is concave upward;
between $E$ and $D$ the bending moment is negative and the beam is concave downward (Fig. 9.5b). We also note that the largest value of the curvature (i.e., the smallest value of the radius of curvature) occurs at the support $C$, where $|M|$ is maximum.

From the information obtained on its curvature, we get a fairly good idea of the shape of the deformed beam. However, the analysis and design of a beam usually require more precise information on the deflection and the slope of the beam at various points. Of particular importance is the knowledge of the maximum deflection of the beam. In the next section Eq. (9.1) will be used to obtain a relation between the deflection $y$ measured at a given point $Q$ on the axis of the beam and the distance $x$ of that point from some fixed origin (Fig. 9.6). The relation obtained is the equation of the elastic curve, i.e., the equation of the curve into which the axis of the beam is transformed under the given loading (Fig. 9.6b).†

9.3 EQUATION OF THE ELASTIC CURVE

We first recall from elementary calculus that the curvature of a plane curve at a point $Q(x,y)$ of the curve can be expressed as

$$\frac{1}{\rho} = \frac{d^2y}{dx^2} \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{3/2} \tag{9.2}$$

where $dy/dx$ and $d^2y/dx^2$ are the first and second derivatives of the function $y(x)$ represented by that curve. But, in the case of the elastic curve of a beam, the slope $dy/dx$ is very small, and its square is negligible compared to unity. We write, therefore,

$$\frac{1}{\rho} = \frac{d^2y}{dx^2} \tag{9.3}$$

Substituting for $1/\rho$ from (9.3) into (9.1), we have

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI} \tag{9.4}$$

†It should be noted that, in this chapter, $y$ represents a vertical displacement, while it was used in previous chapters to represent the distance of a given point in a transverse section from the neutral axis of that section.
The equation obtained is a second-order linear differential equation; it is the governing differential equation for the elastic curve.

The product $EI$ is known as the flexural rigidity and, if it varies along the beam, as in the case of a beam of varying depth, we must express it as a function of $x$ before proceeding to integrate Eq. (9.4). However, in the case of a prismatic beam, which is the case considered here, the flexural rigidity is constant. We may thus multiply both members of Eq. (8.4) by $EI$ and integrate in $x$. We write

$$EI \frac{d^2y}{dx^2} = \int M(x) \, dx + C_1$$

(9.5)

where $C_1$ is a constant of integration. Denoting by $\theta(x)$ the angle, measured in radians, that the tangent to the elastic curve at $Q$ forms with the horizontal (Fig. 9.7), and recalling that this angle is very small, we have

$$\frac{dy}{dx} = \tan \theta = \theta(x)$$

Thus, we write Eq. (9.5) in the alternative form

$$EI \theta(x) = \int M(x) \, dx + C_1$$

(9.5’)

Integrating both members of Eq. (9.5) in $x$, we have

$$EI y = \int \left[ \int M(x) \, dx + C_1 \right] \, dx + C_2$$

$$ EI y = \int_0^x dx \int_0^x M(x) \, dx + C_1 x + C_2$$

(9.6)

where $C_2$ is a second constant, and where the first term in the right-hand member represents the function of $x$ obtained by integrating twice in $x$ the bending moment $M(x)$. If it were not for the fact that the constants $C_1$ and $C_2$ are as yet undetermined, Eq. (9.6) would define the deflection of the beam at any given point $Q$, and Eq. (9.5) or (9.5’) would similarly define the slope of the beam at $Q$.

The constants $C_1$ and $C_2$ are determined from the boundary conditions or, more precisely, from the conditions imposed on the beam by its supports. Limiting our analysis in this section to statically determinate beams, i.e., to beams supported in such a way that the reactions at the supports can be obtained by the methods of statics, we note that only three types of beams need to be considered here (Fig. 9.8): (a) the simply supported beam, (b) the overhanging beam, and (c) the cantilever beam.

In the first two cases, the supports consist of a pin and bracket at $A$ and of a roller at $B$, and require that the deflection be zero at each of these points. Letting $x = x_A$, $y = y_A = 0$ in Eq. (9.6), and then $x = x_B$, $y = y_B = 0$ in the same equation, we obtain two equations that can be solved for $C_1$ and $C_2$. In the case of the cantilever beam (Fig. 9.8c), we note that both the deflection and the slope at $A$ must be zero. Letting $x = x_A$, $y = y_A = 0$ in Eq. (9.6), and $x = x_A$, $\theta = \theta_A = 0$ in Eq. (9.5’), we obtain again two equations that can be solved for $C_1$ and $C_2$. 

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The cantilever beam $AB$ is of uniform cross section and carries a load $P$ at its free end $A$ (Fig. 9.9). Determine the equation of the elastic curve and the deflection and slope at $A$.

**Fig. 9.9**

Using the free-body diagram of the portion $AC$ of the beam (Fig. 9.10), where $C$ is located at a distance $x$ from end $A$, we find

$$M = -Px$$  \hspace{1cm} (9.7)

Substituting for $M$ into Eq. (9.4) and multiplying both members by the constant $EI$, we write

$$EI \frac{d^2y}{dx^2} = -Px$$

Integrating in $x$, we obtain

$$EI \frac{dy}{dx} = -\frac{1}{2}Px^2 + C_1$$  \hspace{1cm} (9.8)

We now observe that at the fixed end $B$ we have $x = L$ and $\theta = dy/dx = 0$ (Fig. 9.11). Substituting these values into (9.8) and solving for $C_1$, we have

$$C_1 = \frac{1}{2}PL^2$$

which we back into (9.8):

$$EI \frac{dy}{dx} = -\frac{1}{2}Px^2 + \frac{1}{2}PL^2$$  \hspace{1cm} (9.9)

Integrating both members of Eq. (9.9), we write

$$EI y = -\frac{1}{6}Px^3 + \frac{1}{2}PL^2x + C_2$$  \hspace{1cm} (9.10)

But, at $B$ we have $x = L, y = 0$. Substituting into (9.10), we have

$$0 = -\frac{1}{6}PL^3 + \frac{1}{2}PL^2 + C_2$$

$$C_2 = -\frac{1}{6}PL^3$$

Carrying the value of $C_2$ back into Eq. (9.10), we obtain the equation of the elastic curve:

$$EI y = -\frac{1}{6}Px^3 + \frac{1}{2}PL^2x - \frac{1}{6}PL^3$$

or

$$y = \frac{P}{6EI}(-x^3 + 3L^2x - 2L^3)$$  \hspace{1cm} (9.11)

The deflection and slope at $A$ are obtained by letting $x = 0$ in Eqs. (9.11) and (9.9). We find

$$y_A = -\frac{PL^3}{3EI} \quad \text{and} \quad \theta_A = \left(\frac{dy}{dx}\right)_A = \frac{PL^2}{2EI}$$
The simply supported prismatic beam $AB$ carries a uniformly distributed load $w$ per unit length (Fig. 9.12). Determine the equation of the elastic curve and the maximum deflection of the beam.

![Fig. 9.12](image)

Drawing the free-body diagram of the portion $AD$ of the beam (Fig. 9.13) and taking moments about $D$, we find that

$$M = \frac{1}{2}wLx - \frac{1}{2}wx^2$$

Substituting for $M$ into Eq. (9.4) and multiplying both members of this equation by the constant $EI$, we write

$$EI \frac{d^2y}{dx^2} = -\frac{1}{2}wx^2 + \frac{1}{2}wLx$$

Integrating twice in $x$, we have

$$EI \frac{dy}{dx} = -\frac{1}{6}wx^3 + \frac{1}{4}wLx^2 + C_1$$

Substituting into Eq. (9.14) the value obtained for $C_1$, we check that the slope of the beam is zero for $x = \frac{L}{2}$ and that the elastic curve has a minimum at the midpoint $C$ of the beam (Fig. 9.15). Letting $x = \frac{L}{2}$ in Eq. (9.16), we have

$$y_C = \frac{w}{24EI} \left( - \frac{L^4}{16} + 2L^3x - \frac{L^2}{2} \right) = -\frac{5wL^4}{384EI}$$

The maximum deflection or, more precisely, the maximum absolute value of the deflection, is thus

$$|y|_{\text{max}} = \frac{5wL^4}{384EI}$$
In each of the two examples considered so far, only one free-body diagram was required to determine the bending moment in the beam. As a result, a single function of $x$ was used to represent $M$ throughout the beam. This, however, is not generally the case. Concentrated loads, reactions at supports, or discontinuities in a distributed load will make it necessary to divide the beam into several portions, and to represent the bending moment by a different function $M(x)$ in each of these portions of beam (Photo 9.1).

Each of the functions $M(x)$ will then lead to a different expression for the slope $u(x)$ and for the deflection $y(x)$. Since each of the expressions obtained for the deflection must contain two constants of integration, a large number of constants will have to be determined. As you will see in the next example, the required additional boundary conditions can be obtained by observing that, while the shear and bending moment can be discontinuous at several points in a beam, the deflection and the slope of the beam cannot be discontinuous at any point.

For the prismatic beam and the loading shown (Fig. 9.16), determine the slope and deflection at point $D$.

We must divide the beam into two portions, $AD$ and $DB$, and determine the function $y(x)$ which defines the elastic curve for each of these portions.

1. From $A$ to $D$ ($x < L/4$). We draw the free-body diagram of a portion of beam $AE$ of length $x < L/4$ (Fig. 9.17). Taking moments about $E$, we have

$$M_1 = \frac{3P}{4}x$$  \hspace{1cm} (9.17)

or, recalling Eq. (9.4),

$$EI \frac{d^2y_1}{dx^2} = \frac{3}{4}Px$$  \hspace{1cm} (9.18)

where $y_1(x)$ is the function which defines the elastic curve for portion $AD$ of the beam. Integrating in $x$, we write

$$EI \theta_1 = EI \frac{dy_1}{dx} = \frac{3}{8}Px^2 + C_1$$  \hspace{1cm} (9.19)

$$EI y_1 = \frac{1}{8}Px^3 + C_1x + C_2$$  \hspace{1cm} (9.20)

2. From $D$ to $B$ ($x > L/4$). We now draw the free-body diagram of a portion of beam $AE$ of length $x > L/4$ (Fig. 9.18) and write

$$M_2 = \frac{3P}{4}x - P\left(x - \frac{L}{4}\right)$$  \hspace{1cm} (9.21)
or, recalling Eq. (9.4) and rearranging terms,
\[ EI \frac{d^2y_2}{dx^2} = -\frac{1}{4}P_x + \frac{1}{4}PL \] (9.22)

where \( y_2(x) \) is the function which defines the elastic curve for portion \( DB \) of the beam. Integrating in \( x \), we write
\[ EI \theta_2 = EI \int \frac{dy_2}{dx} = -\frac{1}{8}P_x^2 + \frac{1}{4}PLx + C_3 \] (9.23)
\[ EI y_2 = -\frac{1}{24}P_x^3 + \frac{1}{8}PLx^2 + C_3x + C_4 \] (9.24)

**Determination of the Constants of Integration.** The conditions that must be satisfied by the constants of integration have been summarized in Fig. 9.19. At the support \( A \), where the deflection is defined by Eq. (9.20), we must have \( x = 0 \) and \( y_1 = 0 \). At the support \( B \), where the deflection is defined by Eq. (9.24), we must have \( x = L \) and \( y_2 = 0 \). Also, the fact that there can be no sudden change in deflection or in slope at point \( D \) requires that \( y_1 = y_2 \) and \( \theta_1 = \theta_2 \) when \( x = L/4 \). We have therefore:
\[ x = 0, y_1 = 0 \], Eq. (9.20): \[ 0 = C_2 \] (9.25)
\[ x = L, y_2 = 0 \], Eq. (9.24): \[ 0 = \frac{1}{12}PL^3 + C_3L + C_4 \] (9.26)
\[ x = L/4, \theta_1 = \theta_2 \], Eqs. (9.19) and (9.23):
\[ \frac{PL^3}{512} + C_1 \frac{L}{4} = \frac{11PL^3}{1536} + C_3 \frac{L}{4} + C_4 \] (9.27)

Solving these equations simultaneously, we find
\[ C_1 = -\frac{7PL^2}{128}, C_2 = 0, C_3 = -\frac{11PL^2}{128}, C_4 = \frac{PL^3}{384} \]

Substituting for \( C_1 \) and \( C_2 \) into Eqs. (9.19) and (9.20), we write that for \( x \leq L/4 \),
\[ EI \theta_1 = \frac{3}{8}P_x^2 - \frac{7PL^2}{128} \] (9.29)
\[ EI y_1 = \frac{1}{8}P_x^3 - \frac{7PL^2}{128} \] (9.30)

Letting \( x = L/4 \) in each of these equations, we find that the slope and deflection at point \( D \) are, respectively,
\[ \theta_D = -\frac{PL^2}{32EI} \quad \text{and} \quad y_D = -\frac{3PL^3}{256EI} \]

We note that, since \( \theta_D \neq 0 \), the deflection at \( D \) is not the maximum deflection of the beam.
We saw in Sec. 9.3 that the equation of the elastic curve can be obtained by integrating twice the differential equation

\[
\frac{d^2y}{dx^2} = \frac{M(x)}{EI} \tag{9.4}
\]

where \(M(x)\) is the bending moment in the beam. We now recall from Sec. 5.3 that, when a beam supports a distributed load \(w(x)\), we have \(dM/dx = V\) and \(dV/dx = -w\) at any point of the beam. Differentiating both members of Eq. (9.4) with respect to \(x\) and assuming \(EI\) to be constant, we have therefore

\[
\frac{d^3y}{dx^3} = \frac{1}{EI} \frac{dM}{dx} = \frac{V(x)}{EI} \tag{9.31}
\]

and, differentiating again,

\[
\frac{d^4y}{dx^4} = \frac{1}{EI} \frac{dV}{dx} = \frac{-w(x)}{EI} \tag{9.32}
\]

We conclude that, when a prismatic beam supports a distributed load \(w(x)\), its elastic curve is governed by the fourth-order linear differential equation

\[
\frac{d^4y}{dx^4} = \frac{w(x)}{EI} \tag{9.32}
\]

Multiplying both members of Eq. (9.32) by the constant \(EI\) and integrating four times, we write

\[
EI \frac{d^4y}{dx^4} = -w(x)
\]

\[
EI \frac{d^3y}{dx^3} = V(x) = -\int w(x)\,dx + C_1
\]

\[
EI \frac{d^2y}{dx^2} = M(x) = -\int dx \int w(x)\,dx + C_1x + C_2 \tag{9.33}
\]

\[
EI \frac{dy}{dx} = EI \theta(x) = -\int dx \int dx \int w(x)\,dx + \frac{1}{2}C_1x^2 + C_2x + C_3
\]

\[
EI y(x) = -\int dx \int dx \int dx \int w(x)\,dx + \frac{1}{6}C_1x^3 + \frac{1}{2}C_2x^2 + C_3x + C_4
\]

The four constants of integration can be determined from the boundary conditions. These conditions include (a) the conditions imposed on the deflection or slope of the beam by its supports (cf. Sec. 9.3), and (b) the condition that \(V\) and \(M\) be zero at the free end of a cantilever beam, or that \(M\) be zero at both ends of a simply supported beam (cf. Sec. 5.3). This has been illustrated in Fig. 9.20.
Deflection of Beams

The method presented here can be used effectively with cantilever or simply supported beams carrying a distributed load. In the case of overhanging beams, however, the reactions at the supports will cause discontinuities in the shear, i.e., in the third derivative of $y$, and different functions would be required to define the elastic curve over the entire beam.

**EXAMPLE 9.04**

The simply supported prismatic beam $AB$ carries a uniformly distributed load $w$ per unit length (Fig. 9.21). Determine the equation of the elastic curve and the maximum deflection of the beam. (This is the same beam and loading as in Example 9.02.)

Since $w = constant$, the first three of Eqs. (9.33) yield

$$EI \frac{d^4y}{dx^4} = -w$$
$$EI \frac{d^3y}{dx^3} = V(x) = -wx + C_1$$
$$EI \frac{d^2y}{dx^2} = M(x) = -\frac{1}{2}wx^2 + C_1x + C_2$$

(9.34)

Noting that the boundary conditions require that $M = 0$ at both ends of the beam (Fig. 9.22), we first let $x = 0$ and $M = 0$ in Eq. (9.34) and obtain $C_2 = 0$. We then make $x = L$ and $M = 0$ in the same equation and obtain $C_1 = \frac{1}{2}wL$.

Carrying the values of $C_1$ and $C_2$ back into Eq. (9.34), and integrating twice, we write

$$EI \frac{d^2y}{dx^2} = -\frac{1}{2}wx^2 + \frac{1}{2}wLx$$
$$EI \frac{dy}{dx} = -\frac{1}{6}wx^3 + \frac{1}{4}wLx^2 + C_3$$
$$EI y = -\frac{1}{24}wx^4 + \frac{1}{12}wLx^3 + C_3x + C_4$$

(9.35)

But the boundary conditions also require that $y = 0$ at both ends of the beam. Letting $x = 0$ and $y = 0$ in Eq. (9.35), we obtain $C_4 = 0$; letting $x = L$ and $y = 0$ in the same equation, we write

$$0 = -\frac{1}{24}wL^4 + \frac{1}{24}wL^4 + C_3L$$
$$C_3 = -\frac{1}{24}wL^3$$

Carrying the values of $C_3$ and $C_4$ back into Eq. (9.35) and dividing both members by $EI$, we obtain the equation of the elastic curve:

$$y = \frac{w}{24EI}(-x^4 + 2Lx^3 - L^3x)$$

(9.36)

The value of the maximum deflection is obtained by making $x = L/2$ in Eq. (9.36). We have

$$|y|_{max} = \frac{5wL^4}{384EI}$$
In the preceding sections, our analysis was limited to statically determinate beams. Consider now the prismatic beam AB (Fig. 9.23a), which has a fixed end at A and is supported by a roller at B. Drawing the free-body diagram of the beam (Fig. 9.23b), we note that the reactions involve four unknowns, while only three equilibrium equations are available, namely
\[ \sum F_x = 0 \quad \sum F_y = 0 \quad \sum M_A = 0 \quad (9.37) \]

Since only \( A_x \) can be determined from these equations, we conclude that the beam is statically indeterminate.

However, we recall from Chaps. 2 and 3 that, in a statically indeterminate problem, the reactions can be obtained by considering the deformations of the structure involved. We should, therefore, proceed with the computation of the slope and deformation along the beam. Following the method used in Sec. 9.3, we first express the bending moment \( M(x) \) at any given point of AB in terms of the distance \( x \) from A, the given load, and the unknown reactions. Integrating in \( x \), we obtain expressions for \( \theta \) and \( y \) which contain two additional unknowns, namely the constants of integration \( C_1 \) and \( C_2 \). But altogether six equations are available to determine the reactions and the constants \( C_1 \) and \( C_2 \); they are the three equilibrium equations (9.37) and the three equations expressing that the boundary conditions are satisfied, i.e., that the slope and deflection at \( A \) are zero, and that the deflection at \( B \) is zero (Fig. 9.24). Thus, the reactions at the supports can be determined, and the equation of the elastic curve can be obtained.

**EXAMPLE 9.05**

Determine the reactions at the supports for the prismatic beam of Fig. 9.23a.

**Equilibrium Equations.** From the free-body diagram of Fig. 9.23b we write
\[ \downarrow \sum F_x = 0: \quad A_x = 0 \]
\[ + \uparrow \sum F_y = 0: \quad A_y + B - wL = 0 \quad (9.38) \]
\[ + \uparrow \sum M_A = 0: \quad M_A + BL - \frac{1}{2}wL^2 = 0 \]
**Equation of Elastic Curve.** Drawing the free-body diagram of a portion of beam \(AC\) (Fig. 9.25), we write

\[
M + \frac{1}{2}wx^2 + MA = 0
\]  

\[
(9.39)
\]

Solving Eq. (9.39) for \(M\) and carrying into Eq. (9.4), we write

\[
EI \frac{d^2y}{dx^2} = -\frac{1}{2}wx^2 + A_y x - MA
\]

Integrating in \(x\), we have

\[
EI \theta = EI \frac{dy}{dx} = -\frac{1}{6}wx^3 + \frac{1}{2}A_y x^2 - MAx + C_1
\]  

\[
(9.40)
\]

\[
EI y = -\frac{1}{24}wx^4 + \frac{1}{6}A_y x^3 - \frac{1}{2}MAx^2 + C_2
\]  

\[
(9.41)
\]

Referring to the boundary conditions indicated in Fig. 9.24, we make \(x = 0, \theta = 0\) in Eq. (9.40), \(x = 0, y = 0\) in Eq. (9.41), and conclude that \(C_1 = C_2 = 0\). Thus, we rewrite Eq. (9.41) as follows:

\[
EI y = -\frac{1}{24}wx^4 + \frac{A_y}{3}x^3 - \frac{1}{2}MAx^2
\]  

\[
(9.42)
\]

But the third boundary condition requires that \(y = 0\) for \(x = L\). Carrying these values into (9.42), we write

\[
0 = -\frac{1}{24}wL^4 + \frac{1}{3}A_y L^3 - \frac{1}{2}MA L^2
\]

or

\[
3MA - A_y L + \frac{1}{2}wL^2 = 0
\]  

\[
(9.43)
\]

Solving this equation along with the three equilibrium equations (9.38), we obtain the reactions at the supports:

\[
A_x = 0 \quad A_y = \frac{3}{2}wL \quad MA = \frac{1}{2}wL^2 \quad B = \frac{3}{2}wL.
\]

In the example we have just considered, there was one redundant reaction, i.e., there was one more reaction than could be determined from the equilibrium equations alone. The corresponding beam is said to be *statically indeterminate to the first degree*. Another example of a beam indeterminate to the first degree is provided in Sample Prob. 9.3. If the beam supports are such that two reactions are redundant (Fig. 9.26a), the beam is said to be *indeterminate to the second degree*. While there are now five unknown reactions (Fig. 9.26b), we find that four equations may be obtained from the boundary conditions (Fig. 9.26c). Thus, altogether seven equations are available to determine the five reactions and the two constants of integration.

**Fig. 9.26** Beam statically indeterminate to the second degree.
**SAMPLE PROBLEM 9.1**

The overhanging steel beam ABC carries a concentrated load \( P \) at end \( C \). For portion \( AB \) of the beam, (a) derive the equation of the elastic curve, (b) determine the maximum deflection, (c) evaluate \( y_{\text{max}} \) for the following data:

- \( W14 \times 68 \)  
- \( I = 722 \) in\(^4\)  
- \( E = 29 \times 10^6 \) psi  
- \( P = 50 \) kips  
- \( L = 15 \) ft  
- \( a = 4 \) ft  
- \( 48 \) in.

**SOLUTION**

**Free-Body Diagrams.** Reactions: \( R_A = Pa/L \downarrow \) \( R_B = P(1 + a/L) \uparrow \)

Using the free-body diagram of the portion of beam \( AD \) of length \( x \), we find

\[
M = -P\frac{a}{L}x \quad (0 < x < L)
\]

**Differential Equation of the Elastic Curve.** We use Eq. (9.4) and write

\[
EI \frac{d^2y}{dx^2} = -P\frac{a}{L}x
\]

Noting that the flexural rigidity \( EI \) is constant, we integrate twice and find

\[
EI \frac{dy}{dx} = -\frac{1}{2}P\frac{a}{L}x^2 + C_1
\]

\[
EI y = -\frac{1}{6}P\frac{a}{L}x^3 + C_1x + C_2
\]

**Determination of Constants.** For the boundary conditions shown, we have

\([x = 0, y = 0]:\) From Eq. (2), we find \( C_2 = 0 \)

\([x = L, y = 0]:\) Again using Eq. (2), we write

\[
EI(0) = -\frac{1}{6}P\frac{a}{L}L^3 + C_1L \quad C_1 = +\frac{1}{6}PaL
\]

\(a. \) **Equation of the Elastic Curve.** Substituting for \( C_1 \) and \( C_2 \) into Eqs. (1) and (2), we have

\[
EI \frac{dy}{dx} = -\frac{1}{2}P\frac{a}{L}x^2 + \frac{1}{6}PaL \quad \frac{dy}{dx} = \frac{PaL}{6EI} \left[ 1 - 3 \left( \frac{x}{L} \right)^2 \right]
\]

\[
EI y = -\frac{1}{6}P\frac{a}{L}x^3 + \frac{1}{6}PaLx \quad y = \frac{PaL^2}{6EI} \left[ \frac{x}{L} - \left( \frac{x}{L} \right)^3 \right]
\]

\(b. \) **Maximum Deflection in Portion \( AB \).** The maximum deflection \( y_{\text{max}} \) occurs at point \( E \) where the slope of the elastic curve is zero. Setting \( dy/dx = 0 \) in Eq. (3), we determine the abscissa \( x_m \) of point \( E \):

\[
0 = \frac{PaL}{6EI} \left[ 1 - 3 \left( \frac{x_m}{L} \right)^2 \right] \quad x_m = \frac{L}{\sqrt{3}} = 0.577L
\]

We substitute \( x_m/L = 0.577 \) into Eq. (4) and have

\[
y_{\text{max}} = \frac{PaL^2}{6EI} \left[ (0.577) - (0.577)^3 \right] \quad y_{\text{max}} = 0.0642 \frac{PaL^2}{EI}
\]

\(c. \) **Evaluation of \( y_{\text{max}} \).** For the data given, the value of \( y_{\text{max}} \) is

\[
(50 \text{ kips})(48 \text{ in.})(180 \text{ in.})^2
\]

\[
y_{\text{max}} = 0.0642 \frac{722 \text{ in}^3}{(29 \times 10^6 \text{ psi})} \quad y_{\text{max}} = 0.238 \text{ in.}
\]
**SAMPLE PROBLEM 9.2**

For the beam and loading shown, determine (a) the equation of the elastic curve, (b) the slope at end A, (c) the maximum deflection.

**SOLUTION**

**Differential Equation of the Elastic Curve.** From Eq. (9.32),

\[
EI \frac{d^4y}{dx^4} = -w(x) = -w_0 \sin \frac{\pi x}{L}
\]  
(1)

Integrate Eq. (1) twice:

\[
EI \frac{d^3y}{dx^3} = V = +w_0 \frac{L^2}{\pi^2} \cos \frac{\pi x}{L} + C_1
\]  
(2)

\[
EI \frac{d^2y}{dx^2} = M = +w_0 \frac{L^2}{\pi^2} \sin \frac{\pi x}{L} + C_1 x + C_2
\]  
(3)

**Boundary Conditions:**

\[
[x = 0, M = 0]: \quad \text{From Eq. (3), we find} \quad C_1 = 0
\]
\[
[x = L, M = 0]: \quad \text{Again using Eq. (3), we write} \quad 0 = w_0 \frac{L^2}{\pi^2} \sin \pi + C_1 L \quad C_1 = 0
\]

Thus:

\[
EI \frac{d^3y}{dx^3} = +w_0 \frac{L^2}{\pi^2} \sin \frac{\pi x}{L}
\]  
(4)

Integrate Eq. (4) twice:

\[
EI \frac{dy}{dx} = EI \theta = -w_0 \frac{L^3}{\pi^3} \cos \frac{\pi x}{L} + C_3
\]  
(5)

\[
EI y = -w_0 \frac{L^4}{\pi^4} \sin \frac{\pi x}{L} + C_3 x + C_4
\]  
(6)

**Boundary Conditions:**

\[
[x = 0, y = 0]: \quad \text{Using Eq. (6), we find} \quad C_4 = 0
\]
\[
[x = L, y = 0]: \quad \text{Again using Eq. (6), we find} \quad C_3 = 0
\]

\[a. \text{ Equation of Elastic Curve} \quad EI y = -w_0 \frac{L^4}{\pi^4} \sin \frac{\pi x}{L}
\]

\[b. \text{ Slope at End A.} \quad \text{For} \ x = 0, \ \text{we have} \quad EI \theta_A = -w_0 \frac{L^3}{\pi^3} \cos 0 \quad \theta_A = \frac{w_0 L^3}{\pi^3 EI}
\]

\[c. \text{ Maximum Deflection.} \quad \text{For} \ x = \frac{L}{2}, \ \text{we have} \quad EI y_{\text{max}} = -w_0 \frac{L^4}{\pi^4} \sin \frac{\pi}{2} \quad y_{\text{max}} = \frac{w_0 L^4}{\pi^4 EI}
\]

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SAMPLE PROBLEM 9.3

For the uniform beam $AB$, (a) determine the reaction at $A$, (b) derive the equation of the elastic curve, (c) determine the slope at $A$. (Note that the beam is statically indeterminate to the first degree.)

SOLUTION

Bending Moment. Using the free body shown, we write

$$+\sum M_D = 0: \quad R_A x - \frac{1}{2} \left( \frac{w_0 x^2}{L} \right) = M = 0 \quad M = R_A x - \frac{w_0 x^3}{6L}$$

Differential Equation of the Elastic Curve. We use Eq. (9.4) and write

$$EI \frac{d^2 y}{dx^2} = R_A x - \frac{w_0 x^3}{6L}$$

Noting that the flexural rigidity $EI$ is constant, we integrate twice and find

$$EI \frac{dy}{dx} = EI \theta = \frac{1}{2} R_A x^2 - \frac{w_0 x^4}{24L} + C_1 \quad (1)$$

$$EI y = \frac{1}{6} R_A x^3 - \frac{w_0 x^5}{120L} + C_1 x + C_2 \quad (2)$$

Boundary Conditions. The three boundary conditions that must be satisfied are shown on the sketch

$[x = 0, y = 0]: \quad C_2 = 0 \quad (3)$

$[x = L, \theta = 0]: \quad \frac{1}{2} R_A L^2 - \frac{w_0 L^3}{24} + C_1 = 0 \quad (4)$

$[x = L, y = 0]: \quad \frac{1}{6} R_A L^3 - \frac{w_0 L^4}{120} + C_1 L + C_2 = 0 \quad (5)$

a. Reaction at $A$. Multiplying Eq. (4) by $L$, subtracting Eq. (5) member by member from the equation obtained, and noting that $C_2 = 0$, we have

$$\frac{1}{3} R_A L^3 - \frac{1}{32} w_0 L^4 = 0 \quad R_A = \frac{1}{12} w_0 L \uparrow$$

We note that the reaction is independent of $E$ and $I$. Substituting $R_A = \frac{1}{12} w_0 L$ into Eq. (4), we have

$$\frac{1}{2} (\frac{1}{6} w_0 L) L^2 - \frac{1}{2} w_0 L^3 + C_1 = 0 \quad C_1 = -\frac{1}{12} w_0 L^3$$

b. Equation of the Elastic Curve. Substituting for $R_A$, $C_1$, and $C_2$ into Eq. (2), we have

$$EI y = \frac{1}{6} \left( \frac{1}{10} w_0 L \right) x^3 - \frac{w_0 x^5}{120L} \left( \frac{1}{120} w_0 L^3 \right) x$$

$$y = -\frac{w_0}{120EI} \left( -x^5 + 2L^3 x^3 - L^4 x \right) \uparrow$$

c. Slope at $A$. We differentiate the above equation with respect to $x$:

$$\theta = \frac{dy}{dx} = \frac{w_0}{120EI} \left( -5x^4 + 6L^2 x^2 - L^4 \right)$$

Making $x = 0$, we have

$$\theta_A = -\frac{w_0 L^3}{120EI} \quad \theta_A = \frac{w_0 L^3}{120EI} \uparrow$$
In the following problems assume that the flexural rigidity $EI$ of each beam is constant.

**9.1 through 9.4** For the loading shown, determine (a) the equation of the elastic curve for the cantilever beam $AB$, (b) the deflection at the free end, (c) the slope at the free end.

**Fig. P9.1**

**Fig. P9.2**

**Fig. P9.3**

**Fig. P9.4**

**9.5 and 9.6** For the cantilever beam and loading shown, determine (a) the equation of the elastic curve for portion $AB$ of the beam, (b) the deflection at $B$, (c) the slope at $B$.

**Fig. P9.5**

**Fig. P9.6**

**9.7** For the beam and loading shown, determine (a) the equation of the elastic curve for portion $AB$ of the beam, (b) the slope at $A$, (c) the slope at $B$.

**Fig. P9.7**
9.8 For the beam and loading shown, determine (a) the equation of the elastic curve for portion AB of the beam, (b) the deflection at midspan, (c) the slope at B.

![Fig. P9.8](image)

9.9 Knowing that beam AB is an S200 × 34 rolled shape and that \( P = 60 \text{ kN}, \ L = 2 \text{ m}, \text{ and } E = 200 \text{ GPa}, \) determine (a) the slope at A, (b) the deflection at C.

9.10 Knowing that beam AB is a W10 × 33 rolled shape and that \( w_0 = 3 \text{ kips/ft}, \ L = 12 \text{ ft}, \text{ and } E = 29 \times 10^6 \text{ psi}, \) determine (a) the slope at A, (b) the deflection at C.

![Fig. P9.10](image)

9.11 (a) Determine the location and magnitude of the maximum deflection of beam AB. (b) Assuming that beam AB is a W360 × 64, \( L = 3.5 \text{ m}, \text{ and } E = 200 \text{ GPa}, \) calculate the maximum allowable value of the applied moment \( M_0 \) if the maximum deflection is not to exceed 1 mm.

![Fig. P9.11](image)

9.12 For the beam and loading shown, (a) express the magnitude and location of the maximum deflection in terms of \( w_0, \ L, \ E, \text{ and } I. \) (b) Calculate the value of the maximum deflection, assuming that beam AB is a W18 × 50 rolled shape and that \( w_0 = 4.5 \text{ kips/ft}, \ L = 18 \text{ ft}, \text{ and } E = 29 \times 10^6 \text{ psi}. \)

![Fig. P9.12](image)
### 9.13 For the beam and loading shown, determine the deflection at point C. Use $E = 29 \times 10^6$ psi.

![Beam Diagram]

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### 9.14 For the beam and loading shown, knowing that $a = 2$ m, $w = 50$ kN/m, and $E = 200$ GPa, determine (a) the slope at support A, (b) the deflection at point C.

![Beam Diagram]

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### 9.15 For the beam and loading shown, determine the deflection at point C. Use $E = 200$ GPa.

![Beam Diagram]

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### 9.16 Knowing that beam $AE$ is an S200 × 27.4 rolled shape and that $P = 17.5$ kN, $L = 2.5$ m, $a = 0.8$ m and $E = 200$ GPa, determine (a) the equation of the elastic curve for portion $BD$, (b) the deflection at the center $C$ of the beam.

![Beam Diagram]
9.17 For the beam and loading shown, determine (a) the equation of the elastic curve, (b) the slope at end A, (c) the deflection at the midpoint of the span.

\[ w = w_0 \left( 1 - \frac{x^2}{L^2} \right) \]

Fig. P9.17

9.18 For the beam and loading shown, determine (a) the equation of the elastic curve, (b) the slope at end A, (c) the deflection at the midpoint of the span.

\[ w = 4w_0 \left( \frac{x}{L} - \frac{x^2}{L^2} \right) \]

Fig. P9.18

9.19 through 9.22 For the beam and loading shown, determine the reaction at the roller support.

9.23 For the beam shown, determine the reaction at the roller support when \( w_0 = 15 \text{ kN/m} \).
9.24 For the beam shown, determine the reaction at the roller support when \( w_0 = 6 \text{ kips/ft}. \)

9.25 through 9.28 Determine the reaction at the roller support and draw the bending moment diagram for the beam and loading shown.

9.29 and 9.30 Determine the reaction at the roller support and the deflection at point \( C. \)

9.31 and 9.32 Determine the reaction at the roller support and the deflection at point \( D \) if \( a \) is equal to \( L/3. \)
**9.33 and 9.34** Determine the reaction at A and draw the bending moment diagram for the beam and loading shown.

*Fig. P9.33 Fig. P9.34*

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**9.6 Using Singularity Functions to Determine the Slope and Deflection of a Beam**

Reviewing the work done so far in this chapter, we note that the integration method provides a convenient and effective way of determining the slope and deflection at any point of a prismatic beam, as long as the bending moment can be represented by a single analytical function $M(x)$. However, when the loading of the beam is such that two different functions are needed to represent the bending moment over the entire length of the beam, as in Example 9.03 (Fig. 9.16), four constants of integration are required, and an equal number of equations, expressing continuity conditions at point $D$, as well as boundary conditions at the supports $A$ and $B$, must be used to determine these constants. If three or more functions were needed to represent the bending moment, additional constants and a corresponding number of additional equations would be required, resulting in rather lengthy computations. Such would be the case for the beam shown in Photo 9.2. In this section these computations will be simplified through the use of the singularity functions discussed in Sec. 5.5.

*Photo 9.2* In this roof structure, each of the joists applies a concentrated load to the beam that supports it.
Let us consider again the beam and loading of Example 9.03 (Fig. 9.16) and draw the free-body diagram of that beam (Fig. 9.27). Using the appropriate singularity function, as explained in Sec. 5.5, to represent the contribution to the shear of the concentrated load $P$, we write

$$V(x) = \frac{3P}{4} - P(x - \frac{1}{4}L)^0$$

Integrating in $x$ and recalling from Sec. 5.5 that in the absence of any concentrated couple, the expression obtained for the bending moment will not contain any constant term, we have

$$M(x) = \frac{3P}{4} x - P(x - \frac{1}{4}L)^2$$

(9.44)

Substituting for $M(x)$ from (9.44) into Eq. (9.4), we write

$$EI \frac{d^2y}{dx^2} = \frac{3P}{4} x - P(x - \frac{1}{4}L)^2$$

(9.45)

and, integrating in $x$,

$$EI \theta = EI \frac{dy}{dx} = \frac{3}{8} Px^2 - \frac{1}{2} P(x - \frac{1}{4}L)^3 + C_1$$

(9.46)

$$EI y = \frac{1}{8} Px^3 - \frac{1}{6} P(x - \frac{1}{4}L)^3 + C_1x + C_2$$

(9.47)†

The constants $C_1$ and $C_2$ can be determined from the boundary conditions shown in Fig. 9.28. Letting $x = 0$, $y = 0$ in Eq. (9.47), we have

$$0 = 0 - \frac{1}{6} P(0 - \frac{1}{4}L)^3 + 0 + C_2$$

which reduces to $C_2 = 0$, since any bracket containing a negative quantity is equal to zero. Letting now $x = L$, $y = 0$, and $C_2 = 0$ in Eq. (9.47), we write

$$0 = \frac{1}{8} PL^3 - \frac{1}{6} P(\frac{3}{4}L)^3 + C_1L$$

Since the quantity between brackets is positive, the brackets can be replaced by ordinary parentheses. Solving for $C_1$, we have

$$C_1 = \frac{7PL^2}{128}$$

We check that the expressions obtained for the constants $C_1$ and $C_2$ are the same that were found earlier in Sec. 9.3. But the need for additional constants $C_3$ and $C_4$ has now been eliminated, and we do not have to write equations expressing that the slope and the deflection are continuous at point $D$.

†The continuity conditions for the slope and deflection at $D$ are “built-in” in Eqs. (9.46) and (9.47). Indeed, the difference between the expressions for the slope $\theta_1$ in $AD$ and the slope $\theta_2$ in $DB$ is represented by the term $-\frac{1}{4}P(x - \frac{1}{4}L)^2$ in Eq. (9.46), and this term is equal to zero at $D$. Similarly, the difference between the expressions for the deflection $y_1$ in $AD$ and the deflection $y_2$ in $DB$ is represented by the term $-\frac{1}{4}P(x - \frac{1}{4}L)^3$ in Eq. (9.47), and this term is also equal to zero at $D$. 

Fig. 9.16 (repeated)

Fig. 9.27 Free-body diagram for beam of Fig. 9.16.

Fig. 9.28 Boundary conditions for beam of Fig. 9.16.
For the beam and loading shown (Fig. 9.29a) and using singularity functions, (a) express the slope and deflection as functions of the distance \( x \) from the support at \( A \), (b) determine the deflection at the midpoint \( D \). Use \( E = 200 \) GPa and \( I = 6.87 \times 10^{-6} \) m\(^4\).

(a) We note that the beam is loaded and supported in the same manner as the beam of Example 5.05. Referring to that example, we recall that the given distributed loading was replaced by the two equivalent open-ended loadings shown in Fig. 9.29b and that the following expressions were obtained for the shear and bending moment:

\[
V(x) = -1.5(x - 0.6)^3 + 1.5(x - 1.8)^3 + 2.6 - 1.2(x - 0.6)^0
\]

\[
M(x) = -0.75(x - 0.6)^2 + 0.75(x - 1.8)^2 + 2.6x - 1.2(x - 0.6)^1 - 1.44(x - 2.6)^0
\]

Integrating the last expression twice, we obtain

\[
EI\theta = -0.25(x - 0.6)^3 + 0.25(x - 1.8)^3 + 1.3x^2 - 0.6(x - 0.6)^2 - 1.44(x - 0.6)^3 + C_1 \tag{9.48}
\]

\[
EIy = -0.0625(x - 0.6)^4 + 0.0625(x - 1.8)^4 + 0.4333(x - 0.6)^3 - 0.2(x - 0.6)^3 - 0.72(x - 2.6)^2 + C_1x + C_2 \tag{9.49}
\]

The constants \( C_1 \) and \( C_2 \) can be determined from the boundary conditions shown in Fig. 9.30. Letting \( x = 0 \), \( y = 0 \) in Eq. (9.49) and noting that all the brackets contain positive quantities and, therefore, are equal to zero, we conclude that \( C_2 = 0 \). Letting now \( x = 3.6 \), \( y = 0 \), and \( C_2 = 0 \) in Eq. (9.49), we write

\[
0 = -0.0625(3.0)^4 + 0.0625(1.8)^4 + 0.4333(3.0)^3 - 0.2(3.0)^3 - 0.72(1.8)^2 + C_1(3.6)
\]

Since all the quantities between brackets are positive, the brackets can be replaced by ordinary parentheses. Solving for \( C_1 \), we find \( C_1 = -2.692 \).

(b) Substituting for \( C_1 \) and \( C_2 \) into Eq. (9.49) and making \( x = x_D = 1.8 \) m, we find that the deflection at point \( D \) is defined by the relation

\[
EIy_D = -0.0625(1.2)^4 + 0.0625(0)^4 + 0.4333(1.8)^3 - 0.2(1.2)^3 - 0.72(-0.8)^2 - 2.692(1.8)
\]

The last bracket contains a negative quantity and, therefore, is equal to zero. All the other brackets contain positive quantities and can be replaced by ordinary parentheses. We have

\[
EIy_D = -0.0625(1.2)^4 + 0.0625(0)^4 + 0.4333(1.8)^3 - 0.2(1.2)^3 - 0 - 2.692(1.8) = -2.794
\]

Recalling the given numerical values of \( E \) and \( I \), we write

\[
(200 \text{ GPa})(6.87 \times 10^{-6} \text{ m}^4)y_D = -2.794 \text{ kN} \cdot \text{m}^3
\]

\[
y_D = -13.64 \times 10^{-3} \text{ m} = -2.03 \text{ mm}
\]
SAMPLE PROBLEM 9.4

For the prismatic beam and loading shown, determine (a) the equation of the elastic curve, (b) the slope at A, (c) the maximum deflection.

SOLUTION

Bending Moment. The equation defining the bending moment of the beam was obtained in Sample Prob. 5.9. Using the modified loading diagram shown, we had [Eq. (3)]:

\[ M(x) = -\frac{w_0}{3L}x^3 + \frac{2w_0}{3L}(x - \frac{L}{2})^3 + \frac{1}{4}w_0Lx \]

a. Equation of the Elastic Curve. Using Eq. (9.4), we write

\[ EI \frac{d^2y}{dx^2} = -\frac{w_0}{3L}x^3 + \frac{2w_0}{3L}(x - \frac{L}{2})^3 + \frac{1}{4}w_0Lx \]

and, integrating twice in \( x \),

\[ EI \theta = -\frac{w_0}{12L}x^4 + \frac{w_0}{6L}(x - \frac{L}{2})^4 + \frac{w_0L}{8}x^2 + C_1 \]

\[ EI y = -\frac{w_0}{60L}x^5 + \frac{w_0}{30L}(x - \frac{L}{2})^5 + \frac{w_0L}{24}x^3 + C_1x + C_2 \]

Boundary Conditions.

[\( x = 0, y = 0 \): Using Eq. (3) and noting that each bracket \( \langle \quad \rangle \) contains a negative quantity and, thus, is equal to zero, we find \( C_2 = 0 \).]

[\( x = L, y = 0 \): Again using Eq. (3), we write

\[ 0 = -\frac{w_0L^4}{60} + \frac{w_0}{30L}(\frac{L}{2})^5 + \frac{w_0L^4}{24} + C_1L \]

\( C_1 = -\frac{5}{192}w_0L^3 \). Substituting \( C_1 \) and \( C_2 \) into Eqs. (2) and (3), we have

\[ EI \theta = -\frac{w_0}{12L}x^4 + \frac{w_0}{6L}(x - \frac{L}{2})^4 + \frac{w_0L}{8}x^2 - \frac{5}{192}w_0L^3 \]

\[ EI y = -\frac{w_0}{60L}x^5 + \frac{w_0}{30L}(x - \frac{L}{2})^5 + \frac{w_0L}{24}x^3 - \frac{5}{192}w_0L^3x \]

b. Slope at A. Substituting \( x = 0 \) into Eq. (4), we find

\[ EI \theta_A = -\frac{5}{192}w_0L^3 \]

\( \theta_A = \frac{5w_0L^3}{192EI} \)

c. Maximum Deflection. Because of the symmetry of the supports and loading, the maximum deflection occurs at point \( C \), where \( x = \frac{L}{2} \). Substituting into Eq. (5), we obtain

\[ EI y_{\max} = w_0L^4 \left[ -\frac{1}{60(32)} + 0 + \frac{1}{24(8)} - \frac{5}{192(2)} \right] = -\frac{w_0L^4}{120} \]

\( y_{\max} = \frac{w_0L^4}{120EI} \)
SAMPLE PROBLEM 9.5

The rigid bar DEF is welded at point D to the uniform steel beam AB. For the loading shown, determine (a) the equation of the elastic curve of the beam, (b) the deflection at the midpoint C of the beam. Use $E = 29 \times 10^6$ psi.

SOLUTION

**Bending Moment.** The equation defining the bending moment of the beam was obtained in Sample Prob. 5.10. Using the modified loading diagram shown and expressing $x$ in feet, we had [Eq. (3)]:

$$M(x) = -25x^2 + 480x - 160(x - 11)^1 - 480(x - 11)^0 \text{ lb \cdot ft}$$

**a. Equation of the Elastic Curve.** Using Eq. (8.4), we write

$$EI(d^2y/dx^2) = -25x^2 + 480x - 160(x - 11)^1 - 480(x - 11)^0 \text{ lb \cdot ft} \quad (1)$$

and, integrating twice in $x$,

$$EI \theta = -8.33 \times 10^3 + 240x^2 - 80(x - 11)^2 - 480(x - 11)^1 + C_1 \text{ lb \cdot ft^2} \quad (2)$$

$$EI y = -2.08x^3 + 80x^3 - 26.67(x - 11)^3 - 240(x - 11)^2 + C_3x + C_2 \text{ lb \cdot ft^3} \quad (3)$$

**Boundary Conditions.**

[x = 0, y = 0]: Using Eq. (3) and noting that each bracket contains a negative quantity and, thus, is equal to zero, we find $C_2 = 0$.

[x = 16 ft, y = 0]: Again using Eq. (3) and noting that each bracket contains a positive quantity and, thus, can be replaced by a parenthesis, we write

$$0 = -2.083(16)^3 + 80(16)^3 - 26.67(5)^3 - 240(5)^2 + C_1(16)$$

$$C_1 = -11.36 \times 10^3$$

Substituting the values found for $C_1$ and $C_2$ into Eq. (3), we have

$$EI y = -2.083x^3 + 80x^3 - 26.67(x - 11)^3 - 240(x - 11)^2 - 11.36 \times 10^3 \text{ lb \cdot ft^3} \quad (3')$$

To determine $EI$, we recall that $E = 29 \times 10^6$ psi and compute

$$I = \frac{1}{12}bh^3 = \frac{1}{12}(1 \text{ in.})(3 \text{ in.})^3 = 2.25 \text{ in}^4$$

$$EI = (29 \times 10^6 \text{ psi})(2.25 \text{ in}^4) = 65.25 \times 10^6 \text{ lb \cdot in}^2$$

However, since all previous computations have been carried out with feet as the unit of length, we write

$$EI = (65.25 \times 10^6 \text{ lb \cdot in}^2)(1 \text{ ft}^2/12 \text{ in}^2) = 453.1 \times 10^3 \text{ lb \cdot ft}^2$$

**b. Deflection at Midpoint C.** Making $x = 8$ ft in Eq. (3’), we write

$$EI y_C = -2.083(8)^3 + 80(8)^3 - 26.67(3)^3 - 240(-3)^2 - 11.36 \times 10^3(8)$$

Noting that each bracket is equal to zero and substituting for $EI$ its numerical value, we have

$$(453.1 \times 10^3 \text{ lb \cdot ft}^2)y_C = -58.45 \times 10^3 \text{ lb \cdot ft}^3$$

and, solving for $y_C$:

$$y_C = -0.1290 \text{ ft} \quad y_C = -1.548 \text{ in.}$$

Note that the deflection obtained is *not* the maximum deflection.
SAMPLE PROBLEM 9.6

For the uniform beam ABC, (a) express the reaction at A in terms of P, L, a, E, and I, (b) determine the reaction at A and the deflection under the load when a = L/2.

SOLUTION

Reactions. For the given vertical load P the reactions are as shown. We note that they are statically indeterminate.

Shear and Bending Moment. Using a step function to represent the contribution of P to the shear, we write

\[ V(x) = R_A - P(x-a) \]

Integrating in x, we obtain the bending moment:

\[ M(x) = R_Ax - P(x-a)^2 \]

Equation of the Elastic Curve. Using Eq. (9.4), we write

\[ EI \frac{dy}{dx} = R_Ax - P(x-a)^2 \]

Integrating twice in x,

\[ EI \frac{dy}{dx} = EI \theta = \frac{1}{2}Ra \frac{x^2}{2} - \frac{1}{2}P(x-a)^2 + C_1 \]

\[ EI y = \frac{1}{6}Ra \frac{x^3}{2} - \frac{1}{6}P(x-a)^3 + C_1 \frac{x}{2} + C_2 \]

Boundary Conditions. Noting that the bracket (x-a) is equal to zero for x = 0, and to (L-a) for x = L, we write

\begin{align*}
[x = 0, y = 0]: & \quad C_2 = 0 \quad \text{(1)} \\
[x = L, y = 0]: & \quad \frac{1}{3}RaL^3 - \frac{1}{6}P(L-a)^3 + C_1L + C_2 = 0 \quad \text{(2)} \\
[x = L, \theta = 0]: & \quad \frac{1}{2}RaL^2 - \frac{1}{2}P(L-a)^2 + C_1 = 0 \quad \text{(3)}
\end{align*}

a. Reaction at A. Multiplying Eq. (2) by L, subtracting Eq. (3) member by member from the equation obtained, and noting that C_2 = 0, we have

\[ \frac{1}{3}RaL^3 - \frac{1}{6}P(L-a)^3 \left[ 3L - (L-a) \right] = 0 \]

\[ R_A = P \left( 1 - \frac{a}{L} \right) \left( 1 + \frac{a}{2L} \right) \]

We note that the reaction is independent of E and I.

b. Reaction at A and Deflection at B when a = \frac{1}{2}L. Making a = \frac{1}{2}L in the expression obtained for R_A, we have

\[ R_A = P(1 - \frac{1}{2})^2(1 + \frac{1}{2}) = 5P/16 \quad R_A = \frac{5}{16} P \]

Substituting a = L/2 and R_A = 5P/16 into Eq. (2) and solving for C_1, we find C_1 = -PL^2/32. Making x = L/2, C_1 = -PL^2/32, and C_2 = 0 in the expression obtained for y, we have

\[ y_B = \frac{7PL^3}{768EI} \quad y_B = \frac{7PL^3}{768EI} \]

Note that the deflection obtained is not the maximum deflection.