Fuzzy Sets and Systems

Lecture 3
(Fuzzy Arithmetic)

Bu Ali Sina University
Computer Engineering Dep.
Spring 2010
Fuzzy Arithmetic
Outline

Fuzzy numbers
Arithmetic operations on intervals
Arithmetic operations on fuzzy numbers
  – Addition
  – Subtraction
  – Multiplication
  – Division
Fuzzy equations
Fuzzy numbers

To represent an inaccurate number we use fuzzy numbers

Example
- About two o’clock
- Around six-thirty
- Approximately six

A number word and a linguistic modifier

Fuzzy number
- A fuzzy set defined in the set of real number
- Degree 1 of central value
- Membership degree decrease from 1 to 0 on both side

In the other word
- Normal fuzzy sets
- The $\alpha$-cuts of fuzzy number are closed intervals
- The support of every fuzzy number is the open interval $(a,d)$
- Convex fuzzy sets
Fuzzy Numbers

The membership function of a fuzzy number is of the form $A: \mathbb{R} \rightarrow [0,1]$

$$A(x) = \begin{cases} 
  f(x) & \text{for } x \in [a,b] \\
  1 & \text{for } x \in [b,c] \\
  g(x) & \text{for } x \in [c,d] \\
  0 & \text{for } x < a \text{ and } x > d
\end{cases}$$
Fuzzy number, Fuzzy interval
Examples of fuzzy numbers

Fuzzy interval

Fuzzy number
Fuzzy numbers of Around 3
Arithmetic operations on intervals

Fuzzy arithmetic is based on two properties of Fuzzy numbers

- Each fuzzy set and also each fuzzy number can fully be represented by its $\alpha$-cuts
- $\alpha$-cuts of each fuzzy number are closed intervals of real numbers for all $\alpha (0,1]$ 

Arithmetic operations on fuzzy numbers are defined in terms of arithmetic operations on their $\alpha$-cuts (on closed interval)
Interval addition (+)

\[[a, b] + [c, d] = [a+c, b+d]\]
Interval subtraction (-)

$$[a, b] - [c, d] = [a-d, b-c]$$
Interval multiplication (. )

\[ [a, b] \cdot [c, d] = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)] \]
Interval division (/)

\[[a, b] / [c, d] = [a, b] \cdot [1/d, 1/c]\]

\=[[\min(a/c, a/d, b/c, b/d), \max(a/c, a/d, b/c, b/d)]\]

\([-1, 1] / [-2, -0.5] = [-2, 2]\]

\([4, 10] / [1, 2] = [2, 10]\)
We consider:

\[
A = [a_1, a_2], \\
B = [b_1, b_2], \\
C = [c_1, c_2], \\
0 = [0, 0], 1 = [1, 1]
\]

- \(A + B = B + A, A \cdot B = B \cdot A\)
- \((A + B) + C = A + (B + C), (A \cdot B) \cdot C = A \cdot (B \cdot C)\)
- \(A = 0 + A = A + 0, A = 1 \cdot A = A \cdot 1\)
- \(A \cdot (B + C) \subseteq (A \cdot B) + (A \cdot C)\)
Arithmetic operations on fuzzy numbers

An example

\[ A(x) \]

\[ B(x) \]

\[ (A+B)(x) \]

\[ (A-B)(x) \]
Arithmetic operations on fuzzy numbers

\[ \alpha(A \ast B) = \alpha A \ast \alpha B \]

\[ A \ast B = \bigcup_{\alpha \in [0,1]} \alpha(A \ast B) \]

Where \( \ast \) represents any of +, -, \cdot and / operations
Fuzzy numbers for illustrating arithmetic operations

\[ A(x) = \begin{cases} 
0 & \text{for } x < -1 \text{ and } x > 3 \\
(x+1)/2 & \text{for } -1 \leq x \leq 1 \\
(3-x)/2 & \text{for } 1 \leq x \leq 3 
\end{cases} \]

\[ B(x) = \begin{cases} 
0 & \text{for } x < 1 \text{ and } x > 5 \\
(x-1)/2 & \text{for } 1 \leq x \leq 3 \\
(5-x)/2 & \text{for } 3 \leq x \leq 5 
\end{cases} \]
Construction of $\alpha$-cuts of $A(x)$

$A(\alpha a_1) = (\alpha a_1 + 1)/2 = \alpha$

$A(\alpha a_2) = (3 - \alpha a_2)/2 = \alpha$

$\alpha A = [\alpha a_1, \alpha a_2]$  
$= [2\alpha - 1, 3 - 2\alpha]$

$A(x) = \begin{cases} 
0 & \text{for } x < -1 \text{ and } x > 3 \\
(x + 1)/2 & \text{for } -1 \leq x \leq 1 \\
(3 - x)/2 & \text{for } 1 \leq x \leq 3
\end{cases}$
Construction of $\alpha$-cuts of $B(x)$

\[
B(\alpha b_1) = (\alpha b_1 - 1)/2 = \alpha \\
B(\alpha b_2) = (5 - \alpha b_2)/2 = \alpha
\]

$\alpha B = [\alpha b_1, \alpha b_2] = [2\alpha + 1, 5 - 2\alpha]$
Fuzzy addition

$\alpha A = [2\alpha - 1, 3 - 2\alpha]$

$\alpha B = [2\alpha + 1, 5 - 2\alpha]$

$\alpha(A + B) = [4\alpha, 8 - 4\alpha]$

$(A + B)(x) = \begin{cases} 
0 & \text{for } x < 0 \text{ and } x > 8 \\
x/4 & \text{for } 0 \leq x \leq 4 \\
(8-x)/4 & \text{for } 4 \leq x \leq 8 
\end{cases}$
Using the extension principle fuzzy addition is defined as

$$\mu_{A+B}(z) = \bigvee (\mu_A(x) \land \mu_B(y))$$

\[ x, y \]

\[ x+y=z \]
Fuzzy Addition Examples

- $A = (3\sim) = 0.3/1 + 0.6/2 + 1/3 + 0.6/4 + 0.3/5$

- $B = (11\sim) = 0.5/10 + 1/11 + 0.5/12$

- $A + B = (0.3^{0.5}/(1+10) + (0.6^{0.5}/(2+10) + (1^{0.5}/(3+10) + (0.6^{0.5}/(4+10) + (0.3^{0.5}/(5+10) + (0.3^{1}/(1+11) + (0.6^{1}/(2+11) + (1^{1}/(3+11) + (0.6^{1}/(4+11) + (0.3^{1}/(5+11) + (0.3^{0.5}/(1+12) + (0.6^{0.5}/(2+12) + (1^{0.5}/(3+12) + (0.6^{0.5}/(4+12) + (0.3^{0.5}/(5+12)$
Fuzzy Addition Examples

- $A = (3\sim) = 0.3/1 + 0.6/2 + 1/3 + 0.6/4 + 0.3/5$
- $B = (11\sim) = 0.5/10 + 1/11 + 0.5/12$
- Getting the minimum of the membership values
  - $A + B = 0.3/11 + 0.5/12 + 0.5/13 + 0.5/14 + 0.3/15 + 0.3/12 + 0.6/13 + 1/14 + 0.6/15 + 0.3/16 + 0.3/13 + 0.5/14 + 0.5/15 + 0.5/16 + 0.3/17$
Fuzzy Addition Examples

- $A = (3\sim) = 0.3/1 + 0.6/2 + 1/3 + 0.6/4 + 0.3/5$
- $B = (11\sim) = 0.5/10 + 1/11 + 0.5/12$

- Getting the maximum of the duplicated values

- $A + B = 0.3/11 + (0.5 \lor 0.3)/12 + (0.5 \lor 0.6 \lor 0.3)/13 + (0.5 \lor 1 \lor 0.5)/14 + (0.3 \lor 0.6 \lor 0.5)/15 + (0.3 \lor 0.5)/16 + 0.3/17$
- $A + B = 0.3/11 + 0.5/12 + 0.6/13 + 1/14 + 0.6/15 + 0.5/16 + 0.3/17$
Fuzzy subtraction

\[ \alpha A = [2\alpha - 1, 3 - 2\alpha] \]
\[ \alpha B = [2\alpha + 1, 5 - 2\alpha] \]
\[ \alpha(A - B) = [4\alpha - 6, 2 - 4\alpha] \]

\[ (A - B)(x) = \begin{cases} 
0 & \text{for } x < -6 \text{ and } x > 2 \\
(x + 6)/4 & \text{for } -6 \leq x \leq -2 \\
(2 - x)/4 & \text{for } -2 \leq x \leq 2 
\end{cases} \]
Fuzzy multiplication

\( a(A \cdot B) = \begin{cases} 
-4\alpha^2 + 12\alpha - 5, & \text{for } \alpha \in (0,0.5] \\
4\alpha^2 - 1, & \text{for } \alpha \in (0.5,1] 
\end{cases} \)

\( (2\alpha-1)(2\alpha+1) \)

\( (3-2\alpha)(5-2\alpha) \)

\( x = -4\alpha^2 + 12\alpha - 5 \)

\( x = 4\alpha^2 - 1 \)

\( x = 4\alpha^2 - 16\alpha + 15 \)

\( (A \cdot B)(x) = \begin{cases} 
0, & \text{for } x < -5 \text{ and } x > 15 
\end{cases} \)

\( [3 - (4 - x)^{1/2}] / 2, \quad \text{for } -5 \leq x < 0 \)

\( (1 + x)^{1/2} / 2, \quad \text{for } 0 \leq x < 3 \)

\( [4 - (1 + x)^{1/2}] / 2, \quad \text{for } 3 \leq x \leq 15 \)
Fuzzy Multiplication Example

\[ A = 2 \approx 2 = \left\{ \frac{0.6}{1} + \frac{1}{2} + \frac{0.8}{3} \right\} \]

\[ B = 6 \approx 6 = \left\{ \frac{0.8}{5} + \frac{1}{6} + \frac{0.7}{7} \right\} \]

\[ A \times B = 12 \approx 12 = \left( \frac{0.6}{1} + \frac{1}{2} + \frac{0.8}{3} \right) \times \left( \frac{0.8}{5} + \frac{1}{6} + \frac{0.7}{7} \right) \]

\[ = \left\{ \frac{\min(0.6,0.8)}{5} + \frac{\min(0.6,1)}{6} + \ldots + \frac{\min(0.8,1)}{18} + \frac{\min(0.8,0.7)}{21} \right\} \]

\[ = \left\{ \frac{0.6}{5} + \frac{0.6}{6} + \frac{0.6}{7} + \frac{0.8}{10} + \frac{1}{12} + \frac{0.7}{14} + \frac{0.8}{15} + \frac{0.8}{18} + \frac{0.7}{21} \right\} \]
Fuzzy Multi. Examples

\[ A = \left\{ \frac{0.2}{1} + \frac{1}{2} + \frac{0.7}{4} \right\} \]

\[ B = \left\{ \frac{0.5}{1} + \frac{1}{2} \right\} \]

If the mapping is not one-to-one, we get:

\[ \mu_{A \times B}(x_1, x_2) = \max_{y = f(x_1, x_2)} \{ \min[\mu_A(x_1), \mu_B(x_2)] \} \]

\[ f(A, B) = A \times B (\text{arithmetic product}) = \left( \frac{0.2}{1} + \frac{1}{2} + \frac{0.7}{4} \right) \times \left( \frac{0.5}{1} + \frac{1}{2} \right) \]

\[ = \left\{ \frac{\min(0.2, 0.5)}{1} + \frac{\max[\min(0.2, 1), \min(0.5, 1)]}{2} \right. \]
\[ + \frac{\max[\min(0.7, 0.5), \min(1, 1)]}{4} + \frac{\min(0.7, 1)}{8} \right\} \]

\[ = \left\{ \frac{0.2}{1} + \frac{0.5}{2} + \frac{1}{4} + \frac{0.7}{8} \right\} \]
Fuzzy division

\[ \alpha(A/B) = \begin{cases} 
\frac{(2\alpha - 1)}{(2\alpha + 1)}, \frac{(3 - 2\alpha)}{(2\alpha + 1)} & \text{for } \alpha \in (0, 0.5] \\
\frac{(2\alpha - 1)}{(5 - 2\alpha)}, \frac{(3 - 2\alpha)}{(2\alpha + 1)} & \text{for } \alpha \in (0.5, 1] 
\end{cases} \]

\[ (A/B)(x) = \begin{cases} 
0 & \text{for } x < -1 \text{ and } x > 3 \\
\frac{(x + 1)}{(2 - 2x)} & \text{for } -1 \leq x < 0 \\
\frac{(5x + 1)}{(2x + 2)} & \text{for } 0 \leq x < 1/3 \\
\frac{(3 - x)}{(2x + 2)} & \text{for } 1/3 \leq x \leq 3 
\end{cases} \]
Here we study two forms of fuzzy equations:

- \(A + x = B\)
- \(A \cdot x = B\) where \(x\) is unknown fuzzy number

\(X = B - A\) is not a correct answer because

\[A + (B - A) = [a_1, a_2] + [b_1 - a_2, b_2 - a_1] = [a_1 + b_1 - a_2, a_2 + b_2 - a_1]\]

Which is not a correct answer.

Let \(X = [x_1, x_2]\) then \(x_1 = b_1 - a_1, x_2 = b_2 - a_2\) and \(x_1 \leq x_2\) thus \(b_1 - a_1 \leq b_2 - a_2\)
In the form of $\alpha \mathcal{A}$ (general form): $^aA + ^aX = ^aB$ and $X = \bigcup_{a \in (0,1)} ^aX$.

(i) $^a b_1 - ^a a_1 \leq ^a b_2 - ^a a_2$ for every $\alpha \in (0, 1]$, and
(ii) $\alpha \leq \beta$ implies $^a b_1 - ^a a_1 \leq ^\beta b_1 - ^\beta a_1 \leq ^\beta b_2 - ^\beta a_2 \leq ^a b_2 - ^a a_2$.

**Example**

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$^aA$</th>
<th>$^aB$</th>
<th>$^aX$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>[4,4]</td>
<td>[6,6]</td>
<td>[2,2]</td>
</tr>
<tr>
<td>0.9</td>
<td>[3,4]</td>
<td>[5,6]</td>
<td>[2,2]</td>
</tr>
<tr>
<td>0.8</td>
<td>[2,4]</td>
<td>[4,6]</td>
<td>[2,2]</td>
</tr>
<tr>
<td>0.7</td>
<td>[2,4]</td>
<td>[3,5]</td>
<td>[1,2]</td>
</tr>
<tr>
<td>0.6</td>
<td>[1,4]</td>
<td>[2,6]</td>
<td>[1,2]</td>
</tr>
<tr>
<td>0.5</td>
<td>[1,5]</td>
<td>[2,7]</td>
<td>[1,2]</td>
</tr>
<tr>
<td>0.4</td>
<td>[1,5]</td>
<td>[2,8]</td>
<td>[1,3]</td>
</tr>
<tr>
<td>0.3</td>
<td>[1,5]</td>
<td>[2,8]</td>
<td>[1,3]</td>
</tr>
<tr>
<td>0.2</td>
<td>[0,5]</td>
<td>[1,9]</td>
<td>[1,4]</td>
</tr>
<tr>
<td>0.1</td>
<td>[0,6]</td>
<td>[0,10]</td>
<td>[0,4]</td>
</tr>
</tbody>
</table>

$A = .2/[0, 1] + .6/[1, 2] + .8/[2, 3] + .9/[3, 4] + 1/4 + .5/[4, 5] + .1/[5, 6]$, $B = .1/[0, 1] + .2/[1, 2] + .6/[2, 3] + .7/[3, 4] + .8/[4, 5] + .9/[5, 6]$ $+ 1/6 + .5/(6, 7] + .4/(7, 8] + .2/(8, 9] + .1/(9, 10]$.

$X = \bigcup_{\alpha \in (0,1)} ^aX = .1/[0, 1] + .7/[1, 2] + 1/2 + .4/(2, 3] + .2/(3, 4]$.
Assignment

4.1, 4.5, 4.6,
\[ A = .2/x_1 + .4/x_2 + .6/x_3 + .8/x_4 + 1/x_5 \]

as our example, let us show how this set can be represented by its \( \alpha \)-cuts.

The given fuzzy set \( A \) is associated with only five distinct \( \alpha \)-cuts, which are defined by the following characteristic functions (viewed here as special membership functions):

\[
\begin{align*}
\mathbf{2}A &= 1/x_1 + 1/x_2 + 1/x_3 + 1/x_4 + 1/x_5, \\
\mathbf{4}A &= 0/x_1 + 1/x_2 + 1/x_3 + 1/x_4 + 1/x_5, \\
\mathbf{6}A &= 0/x_1 + 0/x_2 + 1/x_3 + 1/x_4 + 1/x_5, \\
\mathbf{8}A &= 0/x_1 + 0/x_2 + 0/x_3 + 1/x_4 + 1/x_5, \\
\mathbf{1}A &= 0/x_1 + 0/x_2 + 0/x_3 + 0/x_4 + 1/x_5.
\end{align*}
\]

We now convert each of the \( \alpha \)-cuts to a special fuzzy set, \( \alpha A \), defined for each \( x \in X = \{x_1, x_2, x_3, x_4, x_5\} \) as follows:

\[ \alpha A(x) = \alpha \cdot ^\mathbf{\alpha}A(x). \]  \hspace{1cm} (2.1)

We obtain

\[
\begin{align*}
\mathbf{2}A &= .2/x_1 + .2/x_2 + .2/x_3 + .2/x_4 + .2/x_5, \\
\mathbf{4}A &= 0/x_1 + .4/x_2 + .4/x_3 + .4/x_4 + .4/x_5, \\
\mathbf{6}A &= 0/x_1 + 0/x_2 + .6/x_3 + .6/x_4 + .6/x_5, \\
\mathbf{8}A &= 0/x_1 + 0/x_2 + 0/x_3 + .8/x_4 + .8/x_5, \\
\mathbf{1}A &= 0/x_1 + 0/x_2 + 0/x_3 + 0/x_4 + 1/x_5.
\end{align*}
\]

It is now easy to see that the standard fuzzy union of these five special fuzzy sets is exactly the original fuzzy set \( A \). That is,

\[ A = \mathbf{2}A \cup \mathbf{4}A \cup \mathbf{6}A \cup \mathbf{8}A \cup \mathbf{1}A. \]